A Stochastic Approach to the Valuation of Barrier Options in Heston's Stochastic Volatility Model

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Abstract. In the valuation of continuous barrier options the distribution of the first hitting time plays a substantial role. In general, the derivation of this hitting time distribution poses a mathematically challenging problem for continuous but otherwise arbitrary boundary curves. When considering barrier options in the Heston model the non-linearity of the variance process leads to the complication of a non-linear hitting boundary. Here, we choose a stochastic approach to solve this problem in the reduced Heston framework, when the correlation is zero and foreign and domestic interest rates are equal. In this context one of our main findings involves the proof of the reflection principle for a driftless Itô process with a time-dependent variance. Combining the two results, we derive a closed-form formula for the value of a continuous barrier option. Compared to the existing pricing formula, our solution provides further insight into how the barrier option value in the reduced Heston model is constructed. Extending the results to the general Heston framework, allowing for arbitrary correlation and drift, we obtain approximations for the joint random variables of the Itô process and its maximum, in a weak sense. As a consequence, an approximate formula for pricing barrier options is established. A numerical case study illustrates the agreement in results of our approach with standard finite difference methods.

Keywords: Heston model, barrier options, reflection principle

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1 Introduction and Problem Description

Barrier options are among the most heavily traded financial instruments in the derivatives markets. These contracts are similar to plain vanilla options at maturity, but in contrast have a payoff that also depends on whether the path of the underlying asset hits a pre-specified barrier level throughout its option life. This key characteristic of path-dependence gives rise to the greater challenge of pricing barrier options compared to their plain vanilla counterpart. There exist many types of barrier options whose payoffs are based on more complicated functions of basic knock-in or knock-out events. Most of them can be valued time-efficiently in closed-form under classical Black-Scholes (BS) model assumptions. However, it is well-known that the Black-Scholes model does not capture some important properties of volatility that are observed in financial markets, such as non-constant implied volatility surfaces. In this regard, a natural extension of the BS model is to model volatility as a separate stochastic process to provide an improved description of the price dynamics of the underlying asset. Within this class of models, the Heston model is a widely recognized stochastic volatility model used by practitioners to price plain vanilla and exotic options.

While there exists a closed-form formula for plain vanilla and a few other options in the presence of stochastic volatility, there is no analytic valuation formula for barrier options in the Heston model available and as a consequence practitioners in the financial industry rely mainly on numerical techniques such as finite difference methods or simulation to price barrier options. But even though simulation methods, such as Monte Carlo simulation, the exact simulation method developed by Broadie and Kaya (2006) or the efficient simulation by Andersen (2008) in the Heston model are easy to implement, they are often found to be too slow for the trading environment due to the large number of sample paths and discretization steps required for an accurate outcome for pricing path-dependent products. Whereas finite difference methods may provide the means to compute prices for barrier options in a fast and a generally applicable manner, they might be accompanied by inaccuracies caused by the operating procedures based on the problems of discrete Heston model processes, as they operate on discrete spaces as well. Finite difference methods to price barrier options in the Heston model have been studied by a number of researchers. In particular, Chiarella, Meyer and Kang (2010) developed a method of lines approach to numerically evaluate barrier option prices and Foulon and In't Hout (2006) outline an ADI finite difference method for option pricing in the Heston model with correlation. These studies, as well as the articles of Haentjens and In't Hout (2012) and Chiarella, Meyer and Kang (2009) that discuss finite difference methods for the case where the model parameters do not satisfy the Feller condition, serve as a foundation to establish current best practice benchmark prices of barrier options for our numerical analysis.

Apart from discretization issues, the mentioned numerical methods above typically fail to reveal the connection between the prices obtained in a stochastic volatility model and the corresponding prices in the Black-Scholes model, as pointed out in Drimus (2011), but for practitioners in the financial industry, it may be important to understand what makes a stochastic volatility price different from the Black-Scholes price that often serves as the basic hedging tool. Therefore, a recent focus in the literature has been the development of approximate closed-form option pricing formulas using

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the main features of the Black-Scholes model. In Alos (2012) vanilla option prices in the Heston model are decomposed as the sum of Black-Scholes prices with a volatility parameter equal to the mean-square-root future average volatility, a term due to correlation, and a term due to variance of volatility. In Drimus (2011) vanilla and forward-starting option prices in the Heston model are expanded in terms of Black-Scholes prices and its Greeks. This expansion shows how the convexity in volatility, as measured by the Black-Scholes volga, and the sensitivity of delta with respect to volatility, as measured by the Black-Scholes vanna, impact option prices. Research of this kind facilitates the investigation of the structure of the Heston model and helps the understanding of what is happening within the Heston model beyond Black-Scholes. Hence, analyzing barrier option prices in the Heston model from a Black-Scholes perspective not only provides insights for practitioners to better understand the resulting prices and as well as to benchmark numerical schemes, but also for academics as the problems arising from the properties of the Heston model are often challenging from a mathematical point of view.

In this paper, we introduce a novel method to derive exact and approximating pricing formulas for options with a continuous barrier level under stochastic volatility dynamics using Black-Scholes calculus. To this end, we consider the stochastic volatility model of Heston (1993), which under the risk-neutral measure is characterized by the following system of stochastic differential equations

$$\begin{aligned} \frac{dS_t}{S_t} &= r dt + \sqrt{v_t} dW_t^S \\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^v. \end{aligned} \tag{1}$$

The process $\{S_t\}_{t\geq 0}$ denotes the spot price and the process $\{v_t\}_{t\geq 0}$ represents the instantaneous variance of the logarithmic spot price with initial variance, $v_0 > 0$. As barrier options are liquidly traded in the Foreign Exchange markets, without loss of generality, in the following discussions we focus on the currency market, and assume the underlying asset to be a foreign exchange rate. Therefore, the risk-neutral drift term r of the underlying price process is set to the difference between the domestic and foreign interest rates, $r_d - r_f$. The two Wiener processes, $\{W_t^S\}_{t\geq 0}$ and $\{W_t^v\}_{t\geq 0}$, are correlated with a constant correlation rate ρ, i.e.

$$dW_t^S \, dW_t^v = \rho dt,$$

where $-1 < \rho < 1$. The remaining parameters of the Heston model, which are the long term variance θ, the rate of mean reversion κ and the volatility of variance σ, are assumed to be constant and positive throughout.

Our contribution in this paper is to obtain a (semi-)closed-form valuation formula for continuous barrier options in a reduced Heston framework, as well as approximations for these types of options in the general Heston model. We demonstrate our derivation using the valuation of an up-and-out call option, with payoff given by

$$(S_T - K)^+ \mathbb{1}_{\left\{\max\limits_{0\leq t\leq T} S_t \leq B\right\}}, \tag{2}$$

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where K denotes the strike price and B the knock-out level of the option. With this choice, we treat the pricing of a barrier option that is a particular numerically challenging case, since the up-and-out call is a barrier option with two boundaries at either side in the payoff profile - below the strike it is zero, then between the strike and the barrier it increases linearly, and then above the barrier it drops immediately. At preceding times, the value of the option will increase until the barrier is getting close, and as the barrier approaches, the value drops back to zero as the probability of knocking out dominates over the payoff. The value of an option with payoff (2) is given by its discounted expected payoff function under the risk-neutral measure. As a consequence, when pricing this type of barrier option one is generally confronted with the problem of deriving the joint distribution of the random variables contained in the payoff function, S_T and $\max_{0 \leq t \leq T} S_t$. For the case of discretely monitoring the underlying asset price, this problem can be solved in closed analytical form as shown in Griebsch and Wystup (2010) and prices can be computed efficiently using the Fourier-cosine expansion as shown in Fang and Oosterlee (2011). However, in continuous time, the derivation of the first hitting time in the Heston model poses a mathematically challenging problem.

In the Black-Scholes model, this option pricing problem can be reformulated in terms of the knock-out probabilities of a Brownian motion with drift and its maximum with respect to constant boundaries. This reduces the central issue of the problem to the derivation of the joint density of a Brownian motion with linearly time-dependent drift and its maximum. Hence, due to the constant volatility assumptions of the Black-Scholes model the entire problem is analytically tractable as shown for example in Shreve (2004).

However, in the context of Heston's stochastic volatility model, two difficulties arise in pricing continuous barrier options. Firstly, the underlying price is not only driven by a Brownian motion, but also by a random volatility process. Secondly, since the realized variance path is neither constant nor linear in time, the task is to compute knock-out probabilities with respect to non-linear boundaries. Major parts of the sequel deal with these two challenges, and while the first issue can be tackled by conditioning on the variance path, the determination of the knock-out probabilities require an approximation approach. Although boundary crossing problems arise in many fields of applied mathematics and different approximation methods have been developed and successfully applied (see for example Novikov, Frishling and Kordzakhia (2003) for an application to option pricing), we use an approximation method that is tailored to the general approach pursued in this paper.

This study follows the line of work in developing methods to derive closed-form valuation formulas in the Heston model. For double barrier options Lipton (2001) proposes (semi-)analytical solutions in a reduced model framework. By setting one of the two barriers in his formula equal to a practically improbable value one can therefore apply the result to obtain an approximation of a single barrier option value. For the purpose of pricing barrier options, Lipton formulates the problem as a set of partial differential equations and uses Fourier series to express the prices as an infinite sum of complex expressions. However, for his approach the derivation of the solution is restricted to the case where the correlation and the interest rate difference in the model is equal to zero. The research by Faulhaber (2002) demonstrates this restriction and why an extension of these techniques to the general Heston framework fails. Nevertheless, for this special case, we can directly compare the outcome of our approach with Lipton's formula.

We employ probabilistic methods rather than techniques to solve differential equations and obtain an analytical solution for the reduced Heston framework. In comparison to Lipton's pricing formula, our solution is of a different format and provides insight into how the barrier option value in the Heston model is constructed. Additionally, the techniques used to derive our result provide us with a foundation to extend the pricing of barrier options to the general Heston model. Thereby, our approach to this problem consists of two major parts. One is to distinguish the cases for

(A) $\rho = 0$ and $r_d = r_f$,

(B) $\rho = 0$ and arbitrary r_d and r_f,

(C) $0 \leq |\rho| < 1$ and arbitrary r_d and r_f,

where we build up our solution from case A to case C, having to solve a more complicated problem for each case and developing techniques to do so along the way. Apart form the technical perspective this distinction of cases is also sensible from a practical point of view. For instance, the case B is an important special case of the Heston model, which is often used in a (parametric) stochastic local volatility model. Since the volatility skew in these models is contributed by a displaced diffusion or CEV component, the stochastic volatility is modeled with zero correlation. The derived pricing method for case B in this paper can be used for this model.
The other part of our strategy is to condition on the information of the variance path generated up until maturity. Thereby, we are allowed to treat the variance in the logarithmic spot price process as a deterministic quantity and are thus able to solve the pricing problem. The last step is to reverse this conditioning by accounting for the distribution of the remaining random variables coming from the process v. Our Ansatz is summarized as follows:

I. Solve case A by

  (i) Conditioning on the variance paths until maturity of the option.

  (ii) Under the knowledge of the variance paths, the logarithmic spot price in the Heston model is normally distributed and we can therefore apply similar techniques, drawn from the BS model, for the derivation of the joint density of the logarithmic spot price and its maximum process.

  (iii) With the joint density from the previous step we derive the valuation formula for up-and-out options.

  (iv) Finally, we resolve the conditioning.

II. Solve case B and C by using the result obtained in I and extend it to an approximate pricing formula.

The remainder of this paper is organized as follows: first we describe the conditioning step and its consequences for the pricing problem in section 2. It turns out that certain probabilistic tools are required in order to proceed with the derivation. Hence, in the appendix A.1.1, the proof of the reflection principle for a driftless Itô process with a time-dependent variance is presented. Then, in section 3, the joint density of the logarithmic spot price and its maximum process is derived for case A, as well as approximations for case B and C. Section 4 shows how the valuation and approximation formulas are obtained for all cases under the condition of known variance paths. The conditioning is resolved in section 5, such that the joint distribution of the remaining random variables with respect to the variance process is determined. The paper presents numerical case studies in section 6 that illustrate the agreement between the results of our developed formulas with a finite difference method, and concludes with a summary of our findings in section 7.

2 First Step: Conditioning

In this section, we describe the logarithmic spot price using integral notation incorporating both of the two model definitions in (1) for S and v into one. This leads to a representation which mainly depends on variance values and an Itô integral with respect to a Brownian motion independent of W^S and W^v. On account of this, the conditioning on the information generated by the variance paths gives rise to a normally distributed logarithmic spot price with time-dependent variance. We analyze how that affects the pricing problem of barrier options in the Heston model.

Combining both definitions for v and S in (1), the logarithmic spot price $x_t = \log S_t$ at time t, $0 < t \leq T$, given the values x_0 and v_0, can be written in integral form as the sum of its initial value, a time-dependent drift α and an Itô integral

$$x_t \quad = \quad x_0 + \alpha(t) + \rho_2 \int_0^t \sqrt{v_s} dW_s, \tag{3}$$

where $\rho_2 = \sqrt{1-\rho^2}$ and where the Brownian motion W arises out of the Cholesky decomposition of W^S into the sum of W^v and another independent Brownian motion W. We use the abbreviation α for the drift

$$\alpha(t) \quad = \quad (r_d - r_f)t - \frac{1}{2}\int_0^t v_s\, ds + \frac{\rho}{\sigma}\Big(v_t - v_0 - \kappa\theta t + \kappa \int_0^t v_s\, ds\Big).$$

Since v_t cannot be negative, conditioning on the σ-algebra generated by the variance process up to time t, i.e. on $\mathcal{G}_t^v = \sigma(\{v_s : 0 \leq s \leq t\})$, yields a random variable $x_t^v := X_t|\mathcal{G}_t^v$, where the only random contribution arises from the integrator dW_s of the Itô integral,

$$Y_t \quad = \quad \rho_2 \int_0^t \sqrt{v_s}\, dW_s. \tag{4}$$

In Shreve (2004), it is shown that an Itô integral Y_t of the format above is normally distributed with expected value zero and variance $\rho_2^2\nu^2(t)$, where

$$\nu^2(t) = \int_0^t v_s ds.$$

Hence, the random variable x_t^v is normally distributed with mean $x_0 + \alpha(t)$ and the same variance. Note that, $\alpha(t)$ is a deterministic continuous function of t when conditioned on a particular realization of $\{v_s\}_{0\leq s\leq t}$, but is not differentiable unless $\rho = 0$.

Now, an up-and-out call with a barrier level $B > K$ has the following payoff

$$V_T = \begin{cases} 0 & \text{if} \quad \max\limits_{0\leq t\leq T} S_t \geq B \\ (S_T - K)^+ & \text{if} \quad \max\limits_{0\leq t\leq T} S_t < B. \end{cases} \tag{5}$$

Since the value of the up-and-out call is given by its discounted expected payoff function, using the tower property and inserting the definition of the payoff in equation (5) yields

$$\text{UOC} := V_0 = e^{-r_d T}\mathbb{E}\Big[\underbrace{\mathbb{E}[(S_T - K)^+ \mathbb{1}_{\{\max\limits_{0\leq t\leq T} S_t < B\}} \,|\, \mathcal{G}_T^v]}_{=:\mathcal{E}^v} \Big| \mathcal{F}_0\Big]. \tag{6}$$

We proceed by analyzing the inner expectation $\mathcal{E}^v$ and define the random variables $\hat{M}$ and $\hat{Y}$ as

$$\begin{aligned} \hat{Y}_t &= \alpha(t) + Y_t \\ \hat{M}_t &= \max\{\hat{Y}_s : 0 \leq s \leq t\}. \end{aligned}$$

Then the inner part of the pricing problem of an up-and-out barrier call option in the Heston model reads as

$$\mathcal{E}^v = \mathbb{E}^v\left[(S_0 e^{\hat{Y}_T} - K)\mathbb{1}_{\{\hat{M}_T < b, \hat{Y}_T > k\}}\right], \tag{7}$$

where $k = \ln(K/S_0)$ and $b = \ln(B/S_0)$. $\mathbb{P}^v$ is associated with the conditional probability measure subject to the filtration $\mathcal{G}_T^v$.

We observe that computing the inner expectation $\mathcal{E}^v$ entails at least finding the unknown joint distribution of $(\hat{Y}, \hat{M})$ under the conditional probability measure $\mathbb{P}^v$. In the appendix A.1.1, we derive an important mathematical tool that is required to solve the just described problem, the reflection principle for Y_t and $M_t = \max\{Y_s : 0 \leq s \leq t\}$. In section 3, we make use of this derived reflection principle in order to determine the distribution of the first hitting time of the process $\{Y_t\}_t$ and in turn the joint distribution of (Y_T, M_T) under the conditional filtration $\mathcal{G}_T^v$ and the probability measure $\mathbb{P}^v$. Then, we tackle the problem of finding the joint distribution of the drift included process $(\alpha(T) + Y_T, \hat{M}_T)$.

3 Second Step: Derivation of the Joint Density

The valuation of continuous barrier options depends on the distribution of the first hitting time. In the Heston model, the derivation of this distribution under the knowledge of the variance paths up until maturity is as straightforward as in the Black-Scholes model if $r_d = r_f$ and $\rho = 0$. This case will be discussed in section 3.1 and we follow closely the theorems 3.7.1, 3.7.3 and 7.2.1 given in Shreve (2004). In section 3.2, we develop a (semi-)closed approximation formula for the case $\rho = 0$ but with arbitrary domestic and foreign interest rates. Finally, the general case will be considered in section 3.3. The following proposition will be useful for all of the three cases.

Proposition 1 *For $t > 0$ the joint distribution of (M_t, Y_t) is given by*

$$f_{M,Y}(m,w) = \frac{2(2m-w)}{\sqrt{2\pi}\rho_2^3\nu^3(t)} \exp\left(-\frac{1}{2}\frac{(2m-w)^2}{\rho_2^2\nu^2(t)}\right) \qquad \text{for } w \leq m, m > 0.$$

Proof: By definition, it is

$$\mathbb{P}^v(M_t \geq m, Y_t \leq w) = \int_m^\infty \int_{-\infty}^w f_{Y,M}(u,s)duds$$

and because Y_t is normally distributed we know that

$$\mathbb{P}^v(Y_t \geq 2m - w) = \frac{1}{\sqrt{2\pi}\rho_2\nu(t)} \int_{2m-w}^\infty e^{-\frac{1}{2}\frac{y^2}{\rho_2^2\nu^2(t)}} dy.$$

By the reflection principle in theorem 1 it follows that

$$\int_m^\infty \int_{-\infty}^w f_{Y,M}(u,s)duds = \frac{1}{\sqrt{2\pi}\rho_2\nu(t)} \int_{2m-w}^\infty e^{-\frac{1}{2}\frac{y^2}{\rho_2^2\nu^2(t)}} dy.$$

Differentiation first with respect to m and then with respect to w leads to the above assertion. □

Proposition 1 states the joint density $f_{Y,M}$ of Y and M (without drift). In order to solve the barrier pricing problem, in section 2 we established that one important part of the solution is to find the unknown joint distribution $f_{\hat{Y},\hat{M}}$ of $\hat{Y}$ and $\hat{M}$ (with drift) under the conditional filtration $\mathcal{G}_T^v$ and the probability measure $\mathbb{P}^v$ (see equation (7)). This problem is treated differently for all three cases A, B, C, that were outlined in the introduction.

3.1 Case $r_d = r_f$ and $\rho = 0$:

In the following, under the assumption that $r_d = r_f$ and $\rho = 0$, the joint density for the process $\alpha(t) + Y_t$ and its maximum is derived.

Proposition 2 *The joint density of $(\hat{M}_T, \hat{Y}_T)$ under $\mathbb{P}^v$ is given by*

$$f_{\hat{M},\hat{Y}}(m,w) = \exp\left(-\frac{1}{2}w - \frac{1}{8}\nu^2(T)\right) \frac{2(2m-w)}{\sqrt{2\pi}\nu^3(T)} e^{-\frac{1}{2}\frac{(2m-w)^2}{\nu^2(T)}} \qquad \text{for } w \leq m, m > 0. \tag{8}$$

Proof: Since at time $t = 0$, the process $\hat{Y}_t$ takes on the value 0, we obviously have $\hat{M}_t \geq 0$ and $\hat{M}_t \geq \hat{Y}_t$. Hence, the pair of random variables $(\hat{M}_t, \hat{Y}_t)$ takes values in the set $\{(m, w) : w \leq m, m \geq 0\}$. For $\rho = 0$ the process Y satisfies

$$dY_t = \sqrt{v_t} dW_t,$$

where W_t is a Brownian motion under the probability measure $\mathbb{P}^v$ with zero drift. Since $\alpha(t) = -\frac{1}{2}\int_0^t v_s ds$, we have

$$d\hat{Y}_t = d\alpha(t) + \sqrt{v_t} dW_t = \sqrt{v_t} d\hat{W}_t,$$

where

$$d\hat{W}_t = dW_t + \frac{d\alpha(t)}{\sqrt{v_t}} = dW_t + \gamma(t)dt \quad \text{and} \quad \gamma(t) = -\frac{1}{2}\sqrt{v_t}.$$

Hence, $\hat{W}_t = \int_0^t \gamma(s)ds + W_t$ is a Brownian motion under $\mathbb{P}^v$ with drift $\gamma(t)$. We define the exponential martingale

$$\hat{H}_t = \exp\left(-\int_0^t \gamma(s)dW_s - \frac{1}{2}\int_0^t \gamma^2(s)ds\right) = \exp\left(-\int_0^t \gamma(s)d\hat{W}_s + \frac{1}{2}\int_0^t \gamma^2(s)ds\right)$$

and the new probability measure $\hat{\mathbb{P}}$ by $\hat{\mathbb{P}}(A) = \int_A \hat{H}_T d\mathbb{P}^v$ for all $A \in \mathcal{G}_T^v$. Conditioned on the sigma-algebra generated by the variance paths, Novikov's condition,

$$\mathbb{E}\Big[\exp\Big\{\frac{1}{2}\int_0^T \Big(-\frac{1}{2}v_s\Big)^2 ds\Big\} \;\Big|\; \mathcal{G}_T^v\Big] < \infty,$$

is immediately satisfied, because $+\infty$ is a natural boundary, i.e. cannot be reached by the process v_t in finite time (cf. Borodin and Salminen, Chapter II). According to Girsanov's theorem, $\hat{W}_t$ is a Brownian motion with zero drift under $\hat{\mathbb{P}}$. Proposition 1 gives us now the joint density of $(\hat{M}, \hat{Y})$ under $\hat{\mathbb{P}}$, which is $\hat{f}_{\hat{M},\hat{Y}}$. To work out the density of $(\hat{M}, \hat{Y})$ under $\mathbb{P}^v$ we find that

$$\begin{aligned}
\mathbb{P}^v[\hat{M}_T \leq m, \hat{Y}_T \leq w] &= \mathbb{E}^v\left[\mathbb{1}_{\{\hat{M}_T \leq m, \hat{Y}_T \leq w\}}\right] \\
&= \hat{\mathbb{E}}\left[\exp\left(\int_0^T \gamma(s)d\hat{W}_s - \frac{1}{2}\int_0^T \gamma^2(s)ds\right)\mathbb{1}_{\{\hat{M}_T \leq m, \hat{Y}_T \leq w\}}\right] \\
&= \int_{-\infty}^{w}\int_{-\infty}^{m} \exp\left(-\frac{1}{2}y - \frac{1}{8}\nu^2(T)\right)\hat{f}_{\hat{M}_T,\hat{Y}_T}(x, y)dxdy.
\end{aligned}$$

Therefore, differentiation leads to the density of $(\hat{M}, \hat{Y})$ under $\mathbb{P}^v$ as stated in the assertion. □

3.2 Case ρ=0 and $r_d \neq r_f$:

The same approach as in section 3.1 can be used up to proposition 2, since

$$\alpha(t) \;=\; (r_d - r_f)t - \frac{1}{2}\int_0^t v_s\,ds$$

is still differentiable. However, $\alpha(t)$ is different than in the section before, hence the drift of the new Brownian motion $\hat{W}$ must be newly defined as well,

$$\gamma(s) \;=\; \frac{(r_d - r_f)}{\sqrt{v_s}} - \frac{1}{2}\sqrt{v_s}.$$

Unfortunately, the term $\sqrt{v_s}$ in the denominator prevents us from applying exactly the same techniques as in the case of section 3.1. Thus, we derive an approximation of the joint density in the following. Proceeding as in proposition 2 yields

$$\begin{aligned}\mathbb{P}^v[\hat{M}_T \leq m, \hat{Y}_T \leq w] &= \hat{\mathbb{E}}\Big[\exp\Big(\int_0^T \gamma(s)d\hat{W}_s - \frac{1}{2}\int_0^T \gamma^2(s)ds\Big)\mathbb{1}_{\{\hat{M}_T \leq m, \hat{Y}_T \leq w\}}\Big] \\ &= \hat{\mathbb{E}}\Big[\exp\Big((r_d - r_f)\hat{X}_T - \frac{1}{2}\hat{Y}_T - \frac{1}{2}\int_0^T \gamma^2(s)ds\Big)\mathbb{1}_{\{\hat{M}_T \leq m, \hat{Y}_T \leq w\}}\Big], \quad (9)\end{aligned}$$

with

$$\hat{X}_T \;=\; \int_0^T \frac{1}{\sqrt{v_s}}d\hat{W}_s.$$

The expectation in (9) contains three random variables $\hat{X}_T, \hat{Y}_T$ and $\hat{M}_T$. Therefore, to compute the distribution function of $(\hat{M}_T, \hat{Y}_T)$ under $\mathbb{P}^v$ we need the joint density of $(\hat{X}_T, \hat{Y}_T, \hat{M}_T)$ under $\hat{\mathbb{P}}$. However, it seems difficult to derive this density by means of the reflection principle (as done for the joint density of $(\hat{Y}_T, \hat{M}_T)$ under $\hat{\mathbb{P}}$), and for that reason we decide to express the random variable $\hat{X}_T$ in terms of $\hat{Y}_T$. First, we outline the idea before going into details. Abbreviating

$$r = r_d - r_f, \qquad b_I(t) = \frac{1}{2}\int_0^t \gamma^2(s)ds$$

and using conditional densities and interchanging integrations yields

$$\begin{aligned}&\mathbb{P}^v[\hat{M}_T \leq m, \hat{Y}_T \leq w] \\ &\quad= \int_{\mathbb{R}}\int_0^\infty \exp\left(-\frac{1}{2}y - b_I(T)\right)\mathbb{1}_{\{z \leq m, y \leq w\}}\hat{f}_{\hat{Y},\hat{M}}(y,z)\hat{\mathbb{E}}[e^{r\hat{X}_T} \,|\, \hat{Y}_T = y, \hat{M}_T = z]\,dz\,dy.\end{aligned}$$

A simple reformulation of $\hat{X}_T$ that results in a tractable expression of (9) employs the fact that $(\hat{X}_T, \hat{Y}_T)$ is jointly normally distributed with zero mean and a certain covariance matrix as shown in appendix A.1.2. Then there exists a representation of $\hat{X}_T$ in terms of $\hat{Y}_T$ and some independent (with respect to $\hat{Y}_T$) standard normal random variable U: $\hat{X}_T = \kappa_1 \hat{Y}_T + \kappa_2 U$ (this representation is discussed in the next paragraph). The inner expectation then reads as

$$\hat{\mathbb{E}}[\exp(r\hat{X}_T)\,|\,\hat{Y}_T = y, \hat{M}_T = z] \quad = \quad \exp(r\kappa_1 y)\hat{\mathbb{E}}[\exp(r\kappa_2 U)\,|\,\hat{M}_T = z]. \tag{10}$$

Since we do not know the joint distribution between $\hat{X}$ and $\hat{M}$, we also do not know the joint distribution between U and $\hat{M}$ that is necessary to solve this last conditional expectation. Further extending the regression and including $\hat{M}$ as a regressor does not provide any remedies to overcome the problem above of the unknown dependence structure between U and $\hat{M}$. Although it is possible to extend the subsequent calculations of the next section 4 to solve the inner expectation when the density additionally contains a term in the exponential function that is linear in $\hat{M}$, this approach gives nevertheless rise to two more challenges. Since the dependence between $\hat{X}$ and $\hat{M}$ is not linear, the first challenge is to determine the regression coefficients and the distribution of the residual. A simple Monte Carlo simulation could give us an idea of the magnitude of the covariance between $\hat{X}$ and $\hat{M}$. Secondly, solving the inner expectation (as in section 4) results in an expression that not only contains the random variable $\int v_s ds$, but $\int 1/v_s ds$ as well. Their joint distribution can be determined analytically, but computing the density numerically by Fourier inversion poses a difficult problem as indicated in Hurd and Yi (2008).
To this end, the remaining conditional expected value will be approximated by taking the unconditional expectation, i.e. discarding the information $\hat{M}_T = z$. Intuitively, the random variable $\hat{M}_T$ (which refers to the path of $\hat{Y}_T$ only) should not contribute too much information to the value of $\hat{X}_T$ in excess to the information the random variable $\hat{Y}_T$ is contributing to $\hat{X}_T$. In the following we develop this described approach more formally.

Define ε by the equation $\hat{X}_T = \kappa_0 + \kappa_1 \hat{Y}_T + \varepsilon$, then the goal is to determine κ_0 and κ_1 such that $\hat{\mathbb{E}}[\varepsilon] = 0$ and $\hat{\mathbb{E}}[\varepsilon^2]$ is minimal. Obviously, $\kappa_0 = 0$. With respect to the second constraint, we note that

$$\hat{\mathbb{E}}[\varepsilon^2] \quad = \quad \mathrm{Var}(\hat{X}_T) + \mathrm{Var}(\kappa_1 \hat{Y}_T) - 2\mathrm{Cov}(\hat{X}_T, \kappa_1 \hat{Y}_T) \;=\; \sigma^2_{\hat{X}} + \kappa_1^2 \sigma^2_{\hat{Y}} - 2\kappa_1 \sigma_{\hat{X},\hat{Y}},$$

with $\sigma^2_{\hat{X}} = \int_0^T \frac{1}{v_s} ds$, $\sigma^2_{\hat{Y}} = \int_0^T v_s ds$ and $\sigma^2_{\hat{X},\hat{Y}} = T$ as shown in proposition 3 in the appendix A.1.2. A minimization with respect to κ_1 yields

$$\kappa_1 \quad = \quad \frac{\sigma_{\hat{X},\hat{Y}}}{\sigma^2_{\hat{Y}}} \quad = \quad \frac{T}{\nu^2(T)}.$$

In proposition 4 in the appendix A.1.2, we show that the random variable $(\hat{X}_T, \kappa_1 \hat{Y}_T)$ is normally distributed with zero mean and a certain covariance matrix, and that ε is normal with zero mean and variance $\sigma^2_\varepsilon = \int_0^T \frac{1}{v_s} ds - \frac{T^2}{\nu^2(T)}$. In the spirit of the definition for ν^2, we define

$$\nu^2_{\text{inv}}(T) = \int_0^T \frac{1}{v_s} ds.$$

Since in the notation introduced above $\varepsilon = \kappa_2 U$, we have

$$\kappa_2 \;=\; \sigma_\varepsilon \;=\; \sqrt{\nu^2_{\mathrm{inv}}(T) - \frac{T^2}{\nu^2(T)}}.$$

It is easy to check that $\mathrm{Cov}(\hat{Y}_T, \varepsilon) = 0$, hence U is independent from $\hat{Y}_T$. Consequently, the just derived expression for $\hat{X}_T$ equates to its Cholesky decomposition with respect to $\frac{1}{\nu(T)}\hat{Y}_T$ and an independent standard normal random variable U.

Now, substituting $\kappa_1 \hat{Y}_T + \kappa_2 U$ for $\hat{X}_T$ in (9) yields

$$\begin{aligned}\mathbb{P}^v[\hat{M}_T \le m, \hat{Y}_T \le w] &= \hat{\mathbb{E}}\Big[\exp\Big(r\hat{X}_T - \frac{1}{2}\hat{Y}_T - b_I(T)\Big)\mathbb{1}_{\{\hat{M}_T \le m, \hat{Y}_T \le w\}}\Big] \\ &= \hat{\mathbb{E}}\Big[\exp\Big(r\left(\kappa_1 \hat{Y}_T + \kappa_2 U\right) - \frac{1}{2}\hat{Y}_T - b_I(T)\Big)\mathbb{1}_{\{\hat{M}_T \le m, \hat{Y}_T \le w\}}\Big]\end{aligned}$$

Due to the independence of $\hat{Y}_T$ and U, but ignoring the dependence of $\hat{M}_T$ and U, we get

$$\begin{aligned}\mathbb{P}^v[\hat{M}_T \le m, \hat{Y}_T \le w] &\approx \hat{\mathbb{E}}[\exp(r\kappa_2 U)] \times \hat{\mathbb{E}}\Big[\exp\Big(\Big(r\kappa_1 - \frac{1}{2}\Big)\hat{Y}_T - b_I(T)\Big)\mathbb{1}_{\{\hat{M}_T \le m, \hat{Y}_T \le w\}}\Big] \\ &= e^{\frac{1}{2}\kappa_2^2 r^2}\int_{-\infty}^{w}\int_{-\infty}^{m}\exp\Big(\Big(r\kappa_1 - \frac{1}{2}\Big)y - b_I(T)\Big)\;\hat{f}_{\hat{M}_T,\hat{Y}_T}(x,y)\,dx\,dy.\end{aligned}$$

Therefore, the density of $(\hat{M}, \hat{Y})$ under $\mathbb{P}^v$ is given by

$$\frac{\partial^2}{\partial m \partial w}\mathbb{P}^v[\hat{M}_T \le m, \hat{Y}_T \le w] \;\approx\; \frac{2(2m-w)}{\sqrt{2\pi}\nu^3(T)}e^{\left(r\kappa_1 - \frac{1}{2}\right)w - b_1(T) - \frac{1}{2}\frac{(2m-w)^2}{\nu^2(T)}},$$

where $b_1(T) = \frac{1}{2}\left[(r_d - r_f)^2\frac{T^2}{\nu^2(T)} - (r_d - r_f)T + \frac{1}{4}\nu^2(T)\right]$.

Remark 1 *Another possible approach is to define the Radon-Nikodym derivative by the simpler* $\gamma(t) = -\frac{1}{2}\sqrt{v_t}$ *as in section 3.1, i.e.*

$$\frac{d\mathbb{P}}{d\hat{\mathbb{P}}} \;=\; \exp\left(-\frac{1}{2}\int_0^T \sqrt{v_s}\,d\hat{W}_s - \frac{1}{8}\int_0^T v_s ds\right).$$

As a consequence the maximum process $\hat{M}_t$ *refers to a process* $\hat{Y}_t$ *with non-zero drift* $\frac{(r_d - r_f)}{\sqrt{v_t}}$*. This drift is even non-linear, and to date no closed formula has been found for barrier hitting times of Brownian motions with non-linear drift. A huge amount of literature deals with this problem, since it has important applications in other areas than financial mathematics (e.g. sequential tests in mathematical statistics) and therefore different approximation methods were developed for the calculation of the hitting time distribution (see Drabeck (2005) for an overview). However, these methods require at least several values* v_s *for times* $s \in [0, T]$*, and therefore cause an equally large number of nested integrals when resolving the conditioning on the volatility path.*

3.3 Case $r_d \neq r_f$ and $\rho \neq 0$

Allowing for $\rho \neq 0$ complicates the pricing problem even further compared to the cases before. Again we are looking for a Girsanov's transformation in order to eliminate the drift $\alpha(t)$ of $\hat{Y}$ and to proceed the pricing procedure as in the Black-Scholes case. Though $\alpha(t)$ is still deterministic under $\mathbb{P}^v$ in this general case, $d\alpha(t)$ is not of order dt anymore, as the term v_t appears in the drift which is a realization of the Heston variance process.

However, we decide to follow up on the approximation approach taken in the section before and additionally assume a sufficiently smooth variance path around v_t such that the derivative with respect to t can be taken. The following discussion shows that any differentiable approximation $\bar{v}_t$ for v_t (with $0 \leq t \leq T$) can be chosen to define the probability measure $\hat{\mathbb{P}}$. As long as two approximations coincide at times $t = 0$ and $t = T$ the resulting measures for the joint density, $\hat{f}_{\hat{M},\hat{Y}}$, will be the same. The necessity of resolving the conditioning on the variance path in the end suggests to additionally require that the approximation is equal to the realized variance path at time 0 and the option's maturity T. Then the characteristic function for the random variable $(v_T, \nu^2(T))$ allows a fast calculation of the corresponding density function. At the end of this section, we furthermore justify this particular choice of probability measure by showing that the exercise probability of a call option is valued correctly under this approach.

Recall the logarithmic spot to be $x_t^v = x_0 + \alpha(t) + Y_t$ with the deterministic drift

$$\alpha(t) = (r_d - r_f)t - \frac{1}{2}\int_0^t v_s\,ds + \frac{\rho}{\sigma}\Big(v_t - v_0 - \kappa\theta t + \kappa\int_0^t v_s\,ds\Big).$$

Let $\bar{v}_t$ be any differentiable approximation for v_t (i.e. for which $d\bar{v}_t/dt$ exists), satisfying $\bar{v}_0 = v_0$ and $\bar{v}_T = v_T$. Then, in order to get a representation $\frac{d\alpha(t)}{\rho_2\sqrt{v_t}} = \gamma(t)dt$, we substitute $\bar{v}_t'$ for dv_t/dt, and define $\gamma(t)$ by

$$\begin{aligned}\gamma(t) &= \frac{1}{\rho_2\sqrt{v_t}}\Big\{(r_d - r_f) - \frac{1}{2}v_t + \frac{\rho}{\sigma}\Big(\bar{v}_t' - \kappa\theta + \kappa v_t\Big)\Big\}\\ &= \frac{1}{\rho_2}\Big((r_d - r_f) - \frac{\kappa\theta\rho}{\sigma}\Big)\frac{1}{\sqrt{v_t}} + \frac{1}{\rho_2}\Big(\frac{\kappa\rho}{\sigma} - \frac{1}{2}\Big)\sqrt{v_t} + \frac{1}{\rho_2}\frac{\rho}{\sigma}\frac{\bar{v}_t'}{\sqrt{v_t}}.\end{aligned}$$

We see that γ is composed out of three terms with factor $\sqrt{v_t}$, $1/\sqrt{v_t}$ and with $\frac{\bar{v}_t'}{\sqrt{v_t}}$. A gamma function involving terms of the first two types was treated in the cases 3.1 and 3.2 before. The term with respect to $\frac{\bar{v}_t'}{\sqrt{v_t}}$ is new and we have to determine how it changes the approach taken before.
First, we repeat the definitions for $\hat{Y}$ and $\hat{X}$ and note that we need to take the factor ρ_2 into account of our calculations for this case. For case A and B, the variable ρ_2 disappears due to ρ being equal to zero, but for case C we get

$$\hat{Y}_t = \rho_2\int_0^t \sqrt{v_s}d\hat{W}_s, \quad \text{and} \quad \hat{X}_t = \rho_2\int_0^t \frac{1}{\sqrt{v_s}}d\hat{W}_s.$$

Next we define

$$\hat{Z}_t = \rho_2 \int_0^t \frac{\bar{v}'_s}{\sqrt{v_s}} d\hat{W}_s,$$

and in the spirit of the definitions of $\nu(T)$ and $\nu_{\text{inv}}(T)$, we further define

$$\nu_I = \sqrt{\int_0^T \frac{\bar{v}'^2_s}{v_s} ds}, \quad \nu_{II} = \sqrt{\int_0^T \frac{\bar{v}'_s}{v_s} ds}, \quad \nu' = \sqrt{\int_0^T \bar{v}'_s ds},$$

Accordingly, we assume that these integrals exist. Then the distribution of the random variable $(\hat{X}_T, \hat{Y}_T, \hat{Z}_T)$ is a standard normal one with zero mean and a covariance matrix given in proposition 4 in the appendix A.1.2.
Again, we find constants κ_1 and κ_3 for the Cholesky decomposition such that

$$\hat{X}_T = \kappa_1 \hat{Y}_T + \varepsilon^X \quad \text{and} \quad \hat{Z}_T = \kappa_3 \hat{Y}_T + \varepsilon^Z,$$

with $\mathbb{E}[\varepsilon^X] = \mathbb{E}[\varepsilon^Z] = 0$ and each $\mathbb{E}[(\varepsilon^X)^2]$ and $\mathbb{E}[(\varepsilon^Z)^2]$ is minimal. A straightforward solution found as in section 3.2 yields

$$\kappa_1 = \frac{T}{\nu^2(T)} \quad \text{and} \quad \kappa_3 = \frac{(\nu'(T))^2}{\nu^2(T)} = \frac{v_T - v_0}{\nu^2(T)}.$$

The random variable $(\varepsilon^X, \varepsilon^Z)$ is normally distributed with zero mean and covariance matrix as given in proposition 4. Furthermore, it is independent of $\hat{Y}_T$.

Finally, we approximate the distribution function $\mathbb{P}^v[\hat{M}_T \leq m, \hat{Y}_T \leq w]$. A Girsanov transformation that eliminates the drift $\alpha(T)$ of $\hat{Y}_T$ gives the cumulative distribution function (CDF) for the random variable $(\hat{M}_T, \hat{Y}_T)$ under the $\mathbb{P}^v$-measure,

$$\mathbb{P}^v[\hat{M}_T \leq m, \hat{Y}_T \leq w] = \hat{\mathbb{E}}\Big[\exp\Big\{-\frac{1}{2}\int_0^T \gamma^2(s)ds + \int_0^T \gamma(s)d\hat{W}_s\Big\}\mathbb{1}_{\{\hat{M}_T \leq m, \hat{Y}_T \leq w\}}\Big].$$

Using c_1, c_2 and c_3 to abbreviate

$$c_1 = \Big((r_d - r_f) - \frac{\kappa\theta\rho}{\sigma}\Big), \quad c_2 = \Big(\frac{\kappa\rho}{\sigma} - \frac{1}{2}\Big), \quad \text{and } c_3 = \frac{\rho}{\sigma}, \tag{11}$$

we proceed by computing the two integrals in the exponential function of the CDF. For the deterministic integral the expression becomes

$$\begin{aligned} b_I(T) &= -\frac{1}{2}\int_0^T \gamma^2(s)ds \\ &= -\frac{1}{2\rho_2^2}\Big[c_1^2\nu_{\text{inv}}^2(T) + c_2^2\nu^2(T) + c_3^2\nu_I^2(T) + 2c_1c_2T + 2c_1c_3\nu_{II}^2(T) + 2c_2c_3(v_T - v_0)\Big]. \end{aligned}$$

And the stochastic integral yields

$$\int_0^T \gamma(s) d\hat{W}_s = \frac{c_1}{\rho_2^2}\hat{X}_T + \frac{c_2}{\rho_2^2}\hat{Y}_T + \frac{c_3}{\rho_2^2}\hat{Z}_T.$$

Therefore,

$$\mathbb{P}^v[\hat{M}_T \leq m, \hat{Y}_T \leq w] = e^{b_I(T)}\hat{\mathbb{E}}\left[e^{\left(c_1(\kappa_1\hat{Y}_T+\varepsilon^X)+c_2\hat{Y}_T+c_3(\kappa_3\hat{Y}_T+\varepsilon^Z)\right)/\rho_2^2}\mathbb{1}_{\{\hat{M}_T\leq m, \hat{Y}_T\leq w\}}\right].$$

Discarding the information of $\hat{M}$ on $(\varepsilon^X, \varepsilon^Z)$ gives rise to the approximation. Due to the independence of $(\varepsilon^X, \varepsilon^Z)$ from $\hat{Y}_T$, we obtain

$$\begin{aligned}\mathbb{P}^v[\hat{M}_T \leq m, \hat{Y}_T \leq w] &\approx \hat{\mathbb{E}}\left[\exp\left((c_1\varepsilon^X + c_3\varepsilon^Z)/\rho_2^2\right)\right] \\ &\quad \times e^{b_I(T)}\hat{\mathbb{E}}\left[\exp\left(\hat{Y}_T\left(c_1\kappa_1 + c_2 + c_3\kappa_3\right)/\rho_2^2\right)\mathbb{1}_{\{\hat{M}_T\leq m, \hat{Y}_T\leq w\}}\right] \\ &= e^{b_I(T)+b_{II}(T)}\int_{-\infty}^{w}\int_{-\infty}^{m} e^{y(c_1\kappa_1+c_2+c_3\kappa_3)/\rho_2^2}\hat{f}_{\hat{M}_T,\hat{Y}_T}(x,y)dxdy.\end{aligned}$$

with $b_{II}(T) = \frac{1}{2\rho_2^2}\left[c_1^2\left(\nu_{\text{inv}}^2(T) - \frac{T^2}{\nu^2(T)}\right) + c_3^2\left(\nu_I^2(T) - \frac{(v_T-v_0)^2}{\nu^2(T)}\right) + 2c_1c_3\left(\nu_{II}^2(T) - \frac{(v_T-v_0)T}{\nu^2(T)}\right)\right]$.

Now, the density of $(\hat{M}, \hat{Y})$ under $\mathbb{P}^v$ is given by

$$\begin{aligned}\frac{\partial^2}{\partial m \partial w}\mathbb{P}^v[\hat{M}_T \leq m, \hat{Y}_T \leq w] &\approx \frac{2(2m-w)}{\sqrt{2\pi}\rho_2^3\nu^3(T)}\exp\left(-\frac{1}{2}\frac{(2m-w)^2}{\rho_2^2\nu^2(T)}\right) \\ &\quad \times \exp\left(\frac{c_1\kappa_1 + c_2 + c_3\kappa_3}{\rho_2^2}w - b_2(T)\right)\end{aligned}$$

where $-b_2(T) = b_I(T) + b_{II}(T) = -\frac{1}{2\rho_2^2}\Big[c_1^2\frac{T^2}{\nu^2(T)} + c_3^2\frac{(v_T-v_0)^2}{\nu^2(T)} + 2c_1c_3\frac{(v_T-v_0)T}{\nu^2(T)} + c_2^2\nu^2(T) + 2c_1c_2T + 2c_2c_3(v_T - v_0)\Big]$.

We conclude this section by arguing that the above introduced measure change according to γ is an appropriate choice, since it provides us with the potential to eliminate the drift $\alpha(t)$ of $\hat{Y}$, which allows us to make use of the analytical format of $f_{Y,M}$. Moreover, it features the pleasing property that the exercise probability $\mathbb{P}[S_T > K]$ of a call (which is inherent in the up-and-out barrier call) is priced correctly under this approach. Since x_T^v is a normally distributed random variable, the exact value of $\mathbb{P}[S_T > K]$ under $\mathcal{G}^v$ is equal to

$$\begin{aligned}\mathbb{P}^v[S_T > K] &= \mathbb{P}^v[x_T^v > \ln(K)] \\ &= 1 - N\left(\frac{\ln(K/S) - \alpha(T)}{\rho_2\nu(T)}\right) \\ &= 1 - N\left(\frac{\ln(K/S) - (r_d - r_f)T - \left(\frac{\kappa\rho}{\sigma} - \frac{1}{2}\right)\nu^2 - \frac{\rho}{\sigma}(v_T - v_0) + \frac{\kappa\theta\rho}{\sigma}T}{\rho_2\nu(T)}\right). \quad (12)\end{aligned}$$

On the other hand using the approach developed in this section, we obtain the following

$$\begin{aligned}
\mathbb{P}^v[S_T > K] &= 1 - \mathbb{P}^v[\hat{Y}_T < \ln(K) - x_0] \\
&= 1 - \mathbb{E}^v\left[\mathbb{1}_{\{\hat{Y}_T < \ln(K/S)\}}\right] \\
&= 1 - \hat{\mathbb{E}}\left[\exp\left(-\frac{1}{2}\int_0^T \gamma^2(s)ds + \int_0^T \gamma(s)d\hat{W}\right)\mathbb{1}_{\{\hat{Y}_T < \ln(K/S)\}}\right].
\end{aligned}$$

Under $\hat{\mathbb{P}}$, the Itô-integral $\hat{Y}_T$ is normally distributed with zero mean and variance equal to $\rho_2^2\nu^2(T)$. Thus,

$$\mathbb{P}^v[S_T > K] = 1 - e^{b_I(T)+b_{II}(T)}\int_{-\infty}^{\ln(K/S)} \exp\left(yq/(\rho_2^2\nu^2(T))\right)\hat{f}_{\hat{Y}}(y)dy$$

with density $\hat{f}_{\hat{Y}}$ equal to the normal probability distribution function and

$$\begin{aligned}
q &= c_1T + c_2\nu^2(T) + c_3(v_T - v_0) \\
&= (r_d - r_f)T + \left(\frac{\kappa\rho}{\sigma} - \frac{1}{2}\right)\nu^2(T) + \frac{\rho}{\sigma}(v_T - v_0) - \frac{\kappa\theta\rho}{\sigma}T.
\end{aligned}$$

Some simple algebra and integration by substitution leads to the correct result given in (12). Note that, in this case the complication with respect to the entanglement of the processes $\hat{M}$ and $\hat{X}$ does not arise. Hence, we are able to derive closed-form and exact expressions for the exercise probability $\mathbb{P}^v[S_T > K]$ of the call.

4 Third Step: Up-and-out Call Formula

In this section, we solve the inner expectation of the pricing problem for the continuous barrier option as stated in equation (7) as

$$\mathcal{E}^v = \mathbb{E}\left[(S_0e^{\hat{Y}_T} - K)\mathbb{1}_{\{\hat{M}_T < b, \hat{Y}_T > k\}} \mid \mathcal{G}_T^v\right].$$

Therefore, we use the joint density of the random variables, $(\hat{Y}, \hat{M})$, that appears in the expectation under the conditional probability measure $\mathbb{P}^v$ derived in the previous section 3. We express the joint density in a unified format for the three cases A, B and C:

$$f_{\hat{M},\hat{Y}}(m,w) \cong \frac{2(2m-w)}{\sqrt{2\pi}\rho_2^3\nu^3(T)}e^{-\frac{1}{2}\frac{(2m-w)^2}{\rho_2^2\nu^2(T)} + Fw + G},$$

where

$$F = \begin{cases} -\frac{1}{2} & \text{for case A,} \\ r\kappa_1 - \frac{1}{2} & \text{for case B,} \\ (c_1\kappa_1 + c_2 + c_3\kappa_3)/\rho_2^2 & \text{for case C.} \end{cases} \quad \text{and} \quad G = \begin{cases} -\frac{1}{8}\nu^2(T) & \text{for case A,} \\ -b_1(T) & \text{for case B,} \\ -b_2(T) & \text{for case C.} \end{cases}$$

The quantities r, c_1, c_2, c_3 are dependent on the model parameters only, whereas the κ_1, κ_2, κ_3 and b_1, b_2 are additionally functions of time-integrated variance: ν and v_T. For case A the exact equality applies.

For all three cases, we find the integration limits of the inner expectation, apply integration by substitution, and express the remaining integrals in terms of the cumulative normal distribution function as in the Black-Scholes case (see Shreve (2004)).

4.1 Solving the Inner Expectation

The integration limits of the inner expectation in (7) are given by $\{k \le w \le b, \max(w,0) \le m \le b\}$ provided that $0 < S_0 < B$. It follows that

$$\mathcal{E}^v = \int_k^b \int_{w^+}^b (S_0 e^w - K) \frac{2(2m-w)}{\sqrt{2\pi}\rho_2^3 \nu^3(T)} \exp\left(Fw + G - \frac{1}{2}\frac{(2m-w)^2}{\rho_2^2 \nu^2(T)} \right) dm\, dw.$$

Because

$$\frac{\partial}{\partial m} \exp\left(-\frac{(2m-w)^2}{2\rho_2^2 \nu^2(T)} \right) = -\frac{2(2m-w)}{\rho_2^2 \nu^2(T)} \exp\left(-\frac{(2m-w)^2}{2\rho_2^2 \nu^2(T)} \right)$$

we apply integration by substitution and obtain

$$\begin{aligned} \mathcal{E}^v = & \frac{1}{\sqrt{2\pi}\rho_2 \nu(T)} \int_k^b (S_0 e^w - K) \exp\left(Fw + G - \frac{1}{2}\frac{w^2}{\rho_2^2 \nu^2(T)} \right) dw \\ & - \frac{1}{\sqrt{2\pi}\rho_2 \nu(T)} \int_k^b (S_0 e^w - K) \exp\left(Fw + G - \frac{1}{2}\frac{(2b-w)^2}{\rho_2^2 \nu^2(T)} \right) dw. \end{aligned}$$

We set

$$\begin{aligned} I_{1,x} &= \frac{1}{\sqrt{2\pi}\rho_2 \nu(T)} e^G \int_k^b \exp\left(-\frac{1}{2}\left(-2(F+x)w + \frac{w^2}{\rho_2^2 \nu^2(T)} \right) \right) dw \\ I_{2,x} &= \frac{1}{\sqrt{2\pi}\rho_2 \nu(T)} e^{G - \frac{2b^2}{\rho_2^2 \nu^2(T)}} \int_k^b \exp\left(-\frac{1}{2}\left(\left(-2(F+x) - \frac{4b}{\rho_2^2 \nu^2(T)} \right) w + \frac{w^2}{\rho_2^2 \nu^2(T)} \right) \right) dw, \end{aligned}$$

with x taking on values in $\{0,1\}$ and such that

$$\mathcal{E}^v = S_0 I_{1,1} - K I_{1,0} - S_0 I_{2,1} + K I_{2,0}.$$

It remains to solve four integrals of the form

$$\int_a^b e^{-\frac{1}{2}(cw + fw^2)} dw = e^{\frac{1}{8}\frac{c^2}{f}} \frac{\sqrt{2\pi}}{\sqrt{f}} \left[N\left(\sqrt{f}\left(-a - \frac{1}{2}\frac{c}{f} \right) \right) - N\left(\sqrt{f}\left(-b - \frac{1}{2}\frac{c}{f} \right) \right) \right],$$

for arbitrary constants a, b, c, f and using $N(z) = 1 - N(-z)$. We obtain

$$
\begin{aligned}
I_{1,x} &= \exp\left(\frac{1}{2}(F+x)^2\rho_2^2\nu^2(T) + G\right) \times \\
&\left[N\left(\frac{\ln\left(\frac{S_0}{K}\right) + (F+x)\rho_2^2\nu^2(T)}{\rho_2\nu(T)}\right) - N\left(\frac{\ln\left(\frac{S_0}{B}\right) + (F+x)\rho_2^2\nu^2(T)}{\rho_2\nu(T)}\right)\right] \\
I_{2,x} &= \exp\left(\frac{1}{2}(F+x)^2\rho_2^2\nu^2(T) + G + 2b(F+x)\right) \times \\
&\left[N\left(\frac{\ln\left(\frac{B^2}{S_0K}\right) + (F+x)\rho_2^2\nu^2(T)}{\rho_2\nu(T)}\right) - N\left(\frac{\ln\left(\frac{B}{S_0}\right) + (F+x)\rho_2^2\nu^2(T)}{\rho_2\nu(T)}\right)\right].
\end{aligned}
$$

Finally, the value of an up-and-out call option in the Heston model at time $t = 0$ is given by

$$
\text{UOC} \cong e^{-r_d T}\mathbb{E}\left[S_0 I_{1,1} - K I_{1,0} - S_0 I_{2,1} + K I_{2,0}\right]. \tag{13}
$$

For case A this is an exact equality, for the cases B and C it is an approximation.
Note that, both expressions, $I_{1,\cdot}$ and $I_{2,\cdot}$, remain to be functions of only two types of random variables when resolving the condition on the variance paths. For case A, B and C it is the time-integrated variance $\nu^2(T) = \int_0^T v_s ds$ and for case C it is additionally v_T.

Case A: $r_d = r_f$ and $\rho = 0$

$$
\begin{aligned}
I_{1,1} &= N\left(\frac{\ln\left(\frac{S_0}{K}\right) + \frac{1}{2}\nu^2(T)}{\nu(T)}\right) - N\left(\frac{\ln\left(\frac{S_0}{B}\right) + \frac{1}{2}\nu^2(T)}{\nu(T)}\right) \\
I_{1,0} &= N\left(\frac{\ln\left(\frac{S_0}{K}\right) - \frac{1}{2}\nu^2(T)}{\nu(T)}\right) - N\left(\frac{\ln\left(\frac{S_0}{B}\right) - \frac{1}{2}\nu^2(T)}{\nu(T)}\right) \\
I_{2,1} &= \frac{B}{S_0}N\left(\frac{\ln\left(\frac{B^2}{S_0K}\right) + \frac{1}{2}\nu^2(T)}{\nu(T)}\right) - \frac{B}{S_0}N\left(\frac{\ln\left(\frac{B}{S_0}\right) + \frac{1}{2}\nu^2(T)}{\nu(T)}\right) \\
I_{2,0} &= \frac{S_0}{B}N\left(\frac{\ln\left(\frac{B^2}{S_0K}\right) - \frac{1}{2}\nu^2(T)}{\nu(T)}\right) - \frac{S_0}{B}N\left(\frac{\ln\left(\frac{B}{S_0}\right) - \frac{1}{2}\nu^2(T)}{\nu(T)}\right).
\end{aligned} \tag{14}
$$

Case B: r_d, r_f arbitrary and $\rho = 0$

$$
\begin{aligned}
I_{1,1} &= \exp\left((r_d - r_f)T\right) \times \\
&\left[N\left(\frac{\ln\left(\frac{S_0}{K}\right) + (r_d - r_f)T + \frac{1}{2}\nu^2(T)}{\nu(T)}\right) - N\left(\frac{\ln\left(\frac{S_0}{B}\right) + (r_d - r_f)T + \frac{1}{2}\nu^2(T)}{\nu(T)}\right)\right]
\end{aligned}
$$

$$\begin{aligned}
I_{1,0} &= N\left(\frac{\ln\left(\frac{S_0}{K}\right)+(r_d-r_f)T-\frac{1}{2}\nu^2(T)}{\nu(T)}\right)-N\left(\frac{\ln\left(\frac{S_0}{B}\right)+(r_d-r_f)T-\frac{1}{2}\nu^2(T)}{\nu(T)}\right)\\
I_{2,1} &= \exp\left(\left(2/\nu^2(T)\ln(B/S_0)+1\right)(r_d-r_f)T\right)\times\\
&\frac{B}{S_0}\left[N\left(\frac{\ln\left(\frac{B^2}{S_0K}\right)+(r_d-r_f)T+\frac{1}{2}\nu^2(T)}{\nu(T)}\right)-N\left(\frac{\ln\left(\frac{B}{S_0}\right)+(r_d-r_f)T+\frac{1}{2}\nu^2(T)}{\nu(T)}\right)\right]\\
I_{2,0} &= \exp\left(2/\nu^2(T)\ln(B/S_0)(r_d-r_f)T\right)\times\\
&\frac{S_0}{B}\left[N\left(\frac{\ln\left(\frac{B^2}{S_0K}\right)+(r_d-r_f)T-\frac{1}{2}\nu^2(T)}{\nu(T)}\right)-N\left(\frac{\ln\left(\frac{B}{S_0}\right)+(r_d-r_f)T-\frac{1}{2}\nu^2(T)}{\nu(T)}\right)\right].
\end{aligned} \tag{15}$$

Case C: r_d, r_f arbitrary and ρ arbitrary

$$\begin{aligned}
I_{1,1} &= \exp\left(\frac{1}{2}\rho_2^2\nu^2(T)+q\right)\left[N\left(\frac{\ln\left(\frac{S_0}{K}\right)+q+\rho_2^2\nu^2(T)}{\rho_2\nu(T)}\right)-N\left(\frac{\ln\left(\frac{S_0}{B}\right)+q+\rho_2^2\nu^2(T)}{\rho_2\nu(T)}\right)\right]\\
I_{1,0} &= N\left(\frac{\ln\left(\frac{S_0}{K}\right)+q}{\rho_2\nu(T)}\right)-N\left(\frac{\ln\left(\frac{S_0}{B}\right)+q}{\rho_2\nu(T)}\right)\\
I_{2,1} &= \exp\left(\frac{1}{2}\rho_2^2\nu^2(T)+q\right)\left(\frac{B}{S_0}\right)^{\frac{2q}{\rho_2^2\nu^2(T)}+2}\\
&\quad\times\left[N\left(\frac{\ln\left(\frac{B^2}{S_0K}\right)+q+\rho_2^2\nu^2(T)}{\rho_2\nu(T)}\right)-N\left(\frac{\ln\left(\frac{B}{S_0}\right)+q+\rho_2^2\nu^2(T)}{\rho_2\nu(T)}\right)\right]\\
I_{2,0} &= \left(\frac{B}{S_0}\right)^{\frac{2q}{\rho_2^2\nu^2(T)}}\left[N\left(\frac{\ln\left(\frac{B^2}{S_0K}\right)+q}{\rho_2\nu(T)}\right)-N\left(\frac{\ln\left(\frac{B}{S_0}\right)+q}{\rho_2\nu(T)}\right)\right].
\end{aligned} \tag{16}$$

with $q = c_1T + c_2\nu^2(T) + c_3(v_T - v_0)$.

Hence, the last step in section 5, which deals with resolving the conditioning on the variance paths, pays attention to finding the distributions of the remaining random variables.

5 Fourth Step: Resolve Conditioning

After analytically solving the inner expectation of the pricing problem (6) for barrier options, the last step is to resolve the conditioning with respect to the information given by the variance paths up until maturity. For case A, we found an exact solution for the joint density as well as the inner expectation and therefore obtain an exact valuation formula for barrier options in this step. The final result is compared with an existing pricing formula for double barrier options. For case B and C, we can use the same approach as for case A resulting in an approximating formula.

5.1 Case $\rho = 0$ and $r_d = r_f$

The value of an up-and-out call option in the Heston model at time $t = 0$ in the special case where $\rho = 0$ and $r_d = r_f$ is given by

$$\text{UOC} = e^{-r_d T}\mathbb{E}\left[S_0 I_{1,1} - K I_{1,0} - S_0 I_{2,1} + K I_{2,0}\right].$$

The terms $I_{1,\cdot}$ and $I_{2,\cdot}$ contain the random variable $\nu^2(T) = \int_0^T v_s ds$. In order to solve the outer expectation, the distribution of $\nu^2(T)$ must be determined. As shown in the appendix A.1.3, we can derive the characteristic function of $\nu^2(T)$ as

$$\phi_{\nu^2(T)}(u) = \mathbb{E}\left[e^{iu\nu^2(T)}\right] = \exp\left[A(u)v_0 + B(u)\right],$$

for functions A and B, with $d = \sqrt{\kappa^2 - 2\sigma^2 iu}$ and $e^{\pm} = 1 \pm \exp(-dT)$,

$$A(u) = \frac{2iue^-}{de^+ + \kappa e^-},$$
$$B(u) = \frac{\kappa\theta}{\sigma^2}(\kappa - d)T + \frac{2\kappa\theta}{\sigma^2}\ln\left(\frac{2d}{de^+ + \kappa e^-}\right).$$

Therefore, the density of $\nu^2(T)$ is given by Fourier inversion

$$d_{\nu^2(T)}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ixu}\phi_{\nu^2(T)}(u)du = \frac{1}{\pi}\int_{\mathbb{R}^+}\Re\left(e^{-ixu}\phi_{\nu^2(T)}(u)\right)du. \tag{17}$$

Hence, the value of an up-and-out call can be computed by dealing with $I_{j,\cdot}$, $j = 1, 2$, as functions of $\nu^2(T)$,

$$\text{UOC} = e^{-r_d T}\int_{\mathbb{R}_+}\left[S_0 I_{1,1}(x) - K I_{1,0}(x) - S_0 I_{2,1}(x) + K I_{2,0}(x)\right] \times d_{\nu^2(T)}(x)\,dx, \tag{18}$$

where $I_{j,\cdot}$, for $j = 1, 2$, are defined in (14). In summary, the value of an up-and-out call can be interpreted as its Black-Scholes value with time-integrated variance.

In comparison, Faulhaber (2002) states and Lipton (2001) derives the following formula for double barrier call options with lower barrier L and upper barrier U

$$\text{DBC} = e^{-r_d T}2\sqrt{SK}\sum_{n=1}^{\infty}\sin\left(k_n \ln\frac{S}{L}\right)\varphi(k_n)\frac{(-1)^{n+1}k_n\left(\sqrt{\frac{U}{K}} - \sqrt{\frac{K}{U}}\right) + \sin\left(k_n \ln\frac{L}{K}\right)}{\left(k_n^2 + \frac{1}{4}\right)\ln(U/L)} \tag{19}$$

with $e^{\pm}(k) = 1 \pm e^{-\zeta(k)T}$, $k_n = \pi n / \ln(U/L)$, $\zeta(k) = \sqrt{\kappa^2 + \sigma^2 \left(k^2 + \frac{1}{4}\right)}$ and

$$\begin{aligned} \varphi(k) &= \exp\left(A(k)v_0 + B(k)\right) \\ A(k) &= \frac{-\left(k^2 + \frac{1}{4}\right) e^{-}(k)}{\zeta(k)e^{+}(k) + \kappa e^{-}(k)} \\ B(k) &= \frac{\kappa\theta}{\sigma^2}(\kappa - \zeta(k))T + \frac{2\kappa\theta}{\sigma^2} \ln\left(\frac{2\zeta(k)}{\zeta(k)e^{+}(k) - \kappa e^{-}(k)}\right). \end{aligned}$$

Lipton states that this series becomes more accurate as time to maturity becomes large or the barriers move closer to each other. Using this formula (19) for a barrier L with a value close to zero, we can approximate up-an-out call values. In section 6, we show the numerical agreement in results for both formulas (18) and (19).

The theoretical connection between the representation of the UOC price in (18) and Lipton's formula (19) for a lower barrier close to zero is based on the following background.
The formula (19) is the direct equivalent of the formula for a double barrier call option in the Black-Scholes model with constant volatility σ and $r_d = r_f$:

$$\mathsf{DBC} = e^{-r_d T} 2\sqrt{SK} \sum_{n=1}^{\infty} \varphi^{BS}(k_n) \sin\left(k_n \ln \frac{S}{L}\right) \frac{\sin(k_n \ln \frac{L}{K}) + (-1)^{n+1} k_n \left(\sqrt{\frac{U}{K}} - \sqrt{\frac{K}{U}}\right)}{(k_n^2 + \frac{1}{4}) \ln(U/L)}, \tag{20}$$

with $\varphi^{BS}(k) = \exp\left(-\frac{1}{2}(k^2 + \frac{1}{4})\sigma^2 T\right)$.

In Davydov and Linetsky (2002) the derivation of the pricing formula (20) for the general case ($r_d \neq r_f$) is outlined by using classic Fourier series to express the transition density of a Brownian motion with two absorbing barriers. Alternatively, the transition density can be expressed using a series of normal densities. In turn, that leads to a representation of the double barrier call price as an infinite sum of normal probabilities. The authors in Davydov and Linetsky (2002) refer to Kunitomo and Ikeda (1992). We state the resulting formula for $r_d = r_f$:

$$\begin{aligned} \mathsf{DBC} = & \sum_{n=-\infty}^{\infty} \left(\frac{U}{L}\right)^n [N(d_1) - N(d_2)] - \left(\frac{L}{U}\right)^n \left(\frac{L}{S_0}\right) [N(d_3) - N(d_4)], \\ & - K e^{-r_d T} \sum_{n=-\infty}^{\infty} \left(\frac{L}{U}\right)^n [N(d_1 - \sigma\sqrt{T}) - N(d_2 - \sigma\sqrt{T})] \\ & - \left(\frac{U}{L}\right)^n \left(\frac{S_0}{L}\right) [N(d_3 - \sigma\sqrt{T}) - N(d_4 - \sigma\sqrt{T})], \end{aligned} \tag{21}$$

with

$$d_1 = \frac{\ln \frac{S_0 U^{2n}}{K L^{2n}} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \qquad d_2 = \frac{\ln \frac{S_0 U^{2n-1}}{L^{2n}} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

$$d_3 = \frac{\ln \frac{L^{2n+2}}{S_0 K U^{2n}} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \qquad d_4 = \frac{\ln \frac{L^{2n+2}}{S_0 U^{2n+1}} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}.$$

Hence, the identity between (20) and (21) is a consequence of two representations[1] for the transition density. Davydov and Linetsky Davydov and Linetsky (2002) argue that the equivalence between the two representations can be shown by an application of the Poisson summation formula. Moreover, they point out that the two series have very different numerical convergence properties, which is explained in detail in Schroeder (2000).

5.2 Case $r_d \neq r_f$ and $\rho = 0$

The second case, where the interest rate spread is arbitrary and the correlation is equal to zero, works similarly and results in an approximation formula for the up-and-out call option,

$$\text{UOC} \approx e^{-r_d T} \int_{\mathbb{R}_+} \left[S_0 I_{1,1}(x) - K I_{1,0}(x) - S_0 I_{2,1}(x) + K I_{2,0}(x) \right] \times d_{\nu^2(T)}(x)\, dx, \tag{22}$$

where $I_{j,\cdot}$, for $j = 1, 2$, are defined in (15).

5.3 Case $r_d \neq r_f$ and $\rho \neq 0$

The value of an up-and-out call option in the Heston model in the case where the correlation and the interest rate difference can be chosen arbitrary is approximated by

$$\text{UOC} \approx e^{-r_d T} \int_{\mathbb{R}_+^2} \left[S_0 I_{1,1}(x,y) - K I_{1,0}(x,y) - S_0 I_{2,1}(x,y) + K I_{2,0}(x,y) \right] \times d_{\nu^2(T), \upsilon_T}(x,y)\, dx\, dy, \tag{23}$$

where $I_{j,\cdot}$, for $j = 1, 2$, are defined in (16). The functions $I_{1,\cdot}$ and $I_{2,\cdot}$ contain the random variables $\nu^2(T)$ and υ_T. The joint distribution can be determined by the bivariate characteristic function given in the appendix A.1.3 and again the joint density is obtained by Fourier inversion. We simplify this representation further to a joint density involving only a single Fourier inversion

$$\begin{aligned} d_{\upsilon_T, \nu^2(T)}(x,y) &= \frac{1}{\pi} \exp\left(L\kappa\tau + (v_0 - y)\frac{\kappa}{\sigma^2} \right) \\ &\times \int_{\mathbb{R}_+} \Re\left(\frac{2d}{\sigma^2 e^-} e^{-iux - Ld\tau - (v_0+y)\frac{de^+}{\sigma^2 e^-}} \left(\frac{ye^{d\tau}}{v_0} \right)^{L-\frac{1}{2}} I_{2L-1}\left(\frac{4d}{\sigma^2 e^-} \sqrt{y v_0 e^{-d\tau}} \right) \right) du. \end{aligned} \tag{24}$$

The derivation is given in the appendix A.1.3.

[1] In fact, the authors in Davydov and Linetsky (2002) actually state a third representation of the transition density in terms of an inverse Laplace transform of the resolvent kernel.

6 Numerical Analysis

In this section, we present some numerical results. We compare up-and-out call prices computed with three different methods: the here derived formulas for all cases, a finite difference scheme, and for the special case of zero correlation and equal interest rates the pricing formula developed by Lipton. All computations were carried out using Matlab code.

To demonstrate the performance of our valuation approach, we analyze the results for a given set of model parameters, contract data and market data as shown in table 1. For the case B , we replace

Model parameters	κ	θ	σ	ρ	v_0
	2	0.04	0.25	0	0.04
Contract data	Strike	Barrier	T		
	80-100	105-145	1 year		
Market data	S_0	r_d	r_f		
	100	0.03	0.03		
Finite Differences	# grid points in S		# grid points in v		# grid points in t
	200		200		100
Lipton Formula	Lower barrier		Summation accuracy		
	0.0001		10e-9		

Table 1: Parameters for case A.

the interest rates by values such that their difference is non-zero and similarly, for case C, we change both the correlation and the interest rate difference to be non-zero. The model parameters chosen in table 1 define a Heston model set-up, which has been standard in many papers undertaking numerical studies of stochastic volatility models. Here, the parameters are chosen such that the Feller condition is fulfilled and to begin with, this model description is used to examine numerically the three conceptually different methods. First experiments concern computing and comparing up-and-out call prices for the cases A, B and C. However, the pricing of the up-and-out call is a particular numerically challenging case, since it is a barrier option with two boundaries at either side in the payoff profile - below the strike it is zero, then between the strike and the barrier it increases linearly, and then above the barrier it drops immediately. At preceding times, the value of the option will increase until the barrier is getting close, and as the barrier approaches, the value decreases to zero as the probability of knocking out dominates over the payoff. Further experiments concern the evaluation of up-and-in call prices within a model description that satisfies the Feller condition. In a last step, we study the performance of the various methods for a model parameter set which does not satisfy the Feller condition.

For case A, we compute prices for single barrier options with barrier levels between 105 and 145 using formula (18). We benchmark the results against the outcome of an ADI finite difference method

(FD) as described in Foulon and In't Hout (2006) and given parameters as in table 1. Additionally, we compare our method to the pricing formula for double barrier options by Lipton (2001) using a lower barrier equal to an improbable value. All methods work very efficiently and produce stable results. The results are given in terms of absolute errors on the left hand side of figure 1 and in terms of relative price differences on the right hand side. From figure 1, we see that the barrier prices

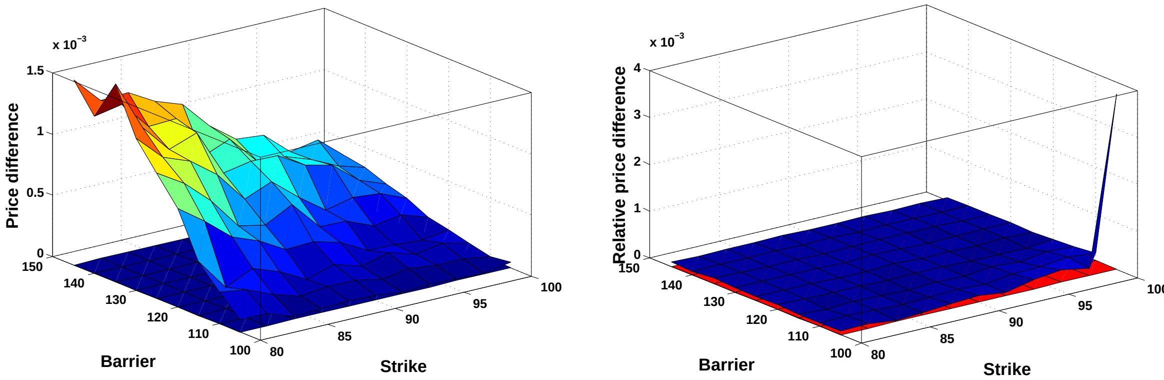

Figure 1: Case A. Left: Up-and-out call option price differences between the Lipton formula and our approach (blue) and between prices obtained by the finite difference method and our approach (coloured). Right: Relative price differences between the Lipton formula and our approach (red) and between prices obtained by the finite difference method and our approach (blue).

generated with our formula are close to those produced with the other two numerical methods. In fact, the prices from both our formula and the one of Lipton match up to a scale of 10^{-7} in absolute terms which we consider to be the reference solution in this case. The approximation with the FD scheme is not as close, but nevertheless as accurate as at least 0.0015 in absolute terms. These deviations might be due to the time discretization that is embedded in the method which implies that the computation yields discretely monitored barrier option prices. The results of relative pricing errors in figure 1 indicate that the greatest deviations in prices between the finite difference method and the other two formulas arise for high strike prices and low barrier levels, which is of the scale of 0.4% and which we need to take into account for the analysis of the other two cases. The computational times per price in Matlab are around 150 seconds for the FD scheme, between 0.35 and 1.1 seconds for formula (18) and around 0.01 seconds for Lipton's formula.

For case B, we evaluate prices for single barrier options for a Heston model with different interest rates. We choose $r_d = 0.05$ and $r_f = 0.02$ and compare absolute and relative errors between the results of formula (22) and the FD scheme. Since Lipton's formula is not valid for this case, we treat the output of the FD scheme as the reference solution keeping in mind that the method shows small numerical inaccuracies to the known exact solution already for case A. The absolute differences between the two methods are at most in the scale of 0.025% of the initial spot price. The relative differences are at most 2.5% in the area of high strike prices and lower barrier levels. For these barrier option prices the FD method showed already deviations of 0.4% to the exact solution. Generally, we identify these errors as a result of discarding the information of the maximum process $\hat{M}$ on the independent part U of $\hat{Y}$, in the derivation of the joint distribution of $\hat{M}$ and $\hat{Y}$. The prices for our example set-up are reported in figure 2. The computational times for both methods remain the same as for case A.

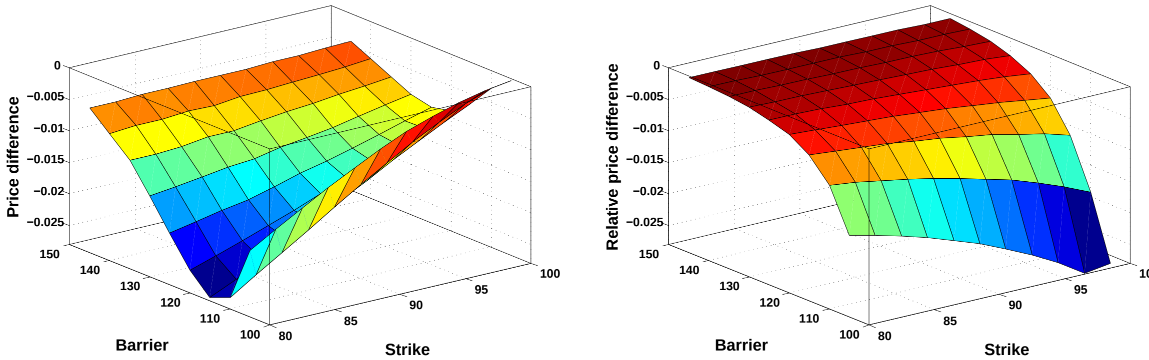

Figure 2: Case B. Differences of absolute and relative up-and-out call option prices between our approach and a finite differences scheme.

For case C, we numerically obtain prices for single barrier options for a model with different interest rates and non-zero correlation. Here, we choose $\rho = -0.5$. All other parameters remain as chosen in case B. The prices for up-and-out call options for different strikes and barrier levels are displayed in figure 3 which shows that prices range between 0 and about 21. The relative differences assist us to relate the size of the absolute errors in perspective to the actual prices. Two methods were used to compute these prices, the method developed in this paper and a finite difference method. We observe that both price surfaces lie very close together and are almost indistinguishable.

The quality of the approximation is shown for the chosen example in figure 4. The absolute differences in this case lie between 0 and 0.25% of the initial spot price, where the peak of the absolute differences arises for barrier options with strikes of 80 and barrier level around $115 - 120$. In terms of relative errors this area exhibits only differences of around 3%, whereas up-and-out calls with low barrier level 105, close to the initial spot price of 100, show the highest relative price differences of 1% up to 8%. However, from figure 3 we see that these options are practically zero in price due to the high probability of being knocked out. Hence, we can conclude that the effect of including a correlation factor to the logarithmic spot price in the model is incorporated numerically in our approximation formula (23) to a good extent. Generally, deviations from the true price are contributed to the following two aspects: the approximation of v_t by a differentiable function and the related measure change, and the error of type "case B" arising from discarding the conditioning on $\hat{M}$. The latter is numerically guided by the coefficients of ε^X and ε^Z, which we call 'X-factor' and 'Z-factor', respectively. For the above analysis, the X-factor and the Z-factor yield a magnitude of both of 0 for case A, of 0.03 and 0 for case B and 0.2533 and -2.6667 for case C, respectively. In table 2 we list a number of test cases that were evaluated in the following. The parameters of table 1 were altered such that either the

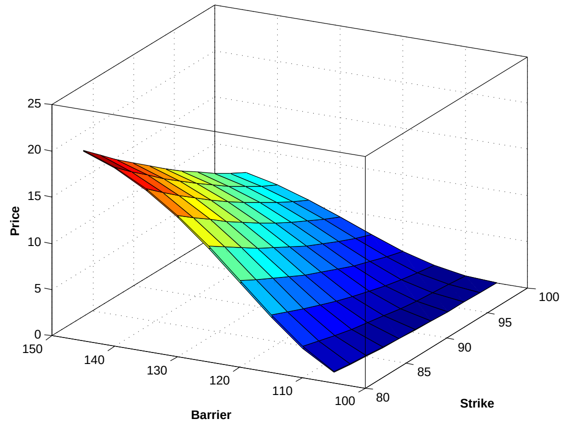

Figure 3: Case C. Up-and-out call option prices computed using our approach and a finite differences scheme.

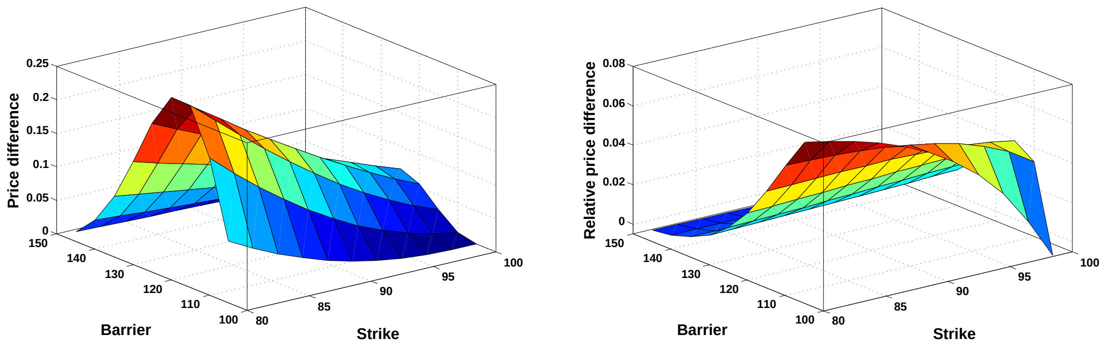

Figure 4: Case C. Differences of absolute and relative up-and-out call option prices between our approach and a finite differences scheme.

X-factor or the Z-factor de- or increases thereby still satisfying the Feller condition. Accordingly, we observe a response in the UOC prices and examine absolute prices differences to the FD scheme that are displayed in figure 5. Additional experiments reveal that a decrease of the coefficients leads to an improvement of the approximation and an increase to a decline in the quality of the approximation of UOC prices. A next observation is that this feature does not persist in the error behavior of the relative price differences. The computational time for case C mainly depends on the computation of the bivariate density $d_{\nu^2(T),v_T}(x,y)$. Since the density does not depend on initial spot price, strike price or barrier level, it can be pre-computed on a given discrete grid and cached. Using the numerical integration tools provided by Matlab we observe computational times between 1 and 8 seconds.

In the before established manner, approximation formulas for the up-and-in call (UIC) with $S_0 < B$

r_d	r_f	κ	θ	σ	ρ	v_0	X-factor	Z-factor
0.05	0.02	2	0.04	0.25	-0.5	0.04	0.253	-2.667
0.01	0.09	1	0.04	0.25	-0.5	0.04	0.0	-2.667
0.05	0.02	5	0.1	0.25	-0.5	0.04	1.373	-2.667
0.05	0.02	4	0.04	0.50	-0.5	0.04	0.253	-1.333
0.05	0.02	0.4	0.04	0.05	-0.5	0.04	0.253	-13.333

Table 2: Heston model parameter sets that yield different X- and Z-factors. All sets satisfy the Feller condition.

and $K < B$ can be derived to be

$$\begin{aligned} \text{UIC} \cong\ & e^{-r_d T} \int_{\mathbb{R}_+^2} \Bigg[S_0 I_{2,1}(x,y) - K I_{2,0}(x,y) + S_0 e^{\frac{1}{2}\rho_2^2 x + q(x,y)} N\left(\frac{\ln\left(\frac{S_0}{B}\right) + q(x,y) + \rho_2^2 x}{\rho_2 \sqrt{x}} \right) \\ & -KN\left(\frac{\ln\left(\frac{S_0}{B}\right) + q(x,y)}{\rho_2\sqrt{x}} \right) \Bigg] \times d_{\nu^2(T),v_T}(x,y)\, dx\, dy. \qquad (25) \end{aligned}$$

Moreover, it can be shown that the property UOC + UIC = Call is analytically fulfilled for the sum of the up-and-out call in (23) and the up-and-in call (25). That means, that any inaccuracies in the approximation of the up-and-out call price are also given in the approximation of the up-and-in call but with the opposite sign. The left hand side of figure 6 clearly shows that the UIC prices computed using (25), but adjusted by the UOC inaccuracies, yield the same result up to an absolute error of 10^{-5} as those obtained with a FD scheme. Hence, the two price surfaces computed with our approach and a finite differences scheme that are shown in the right hand side of figure 6 are visibly indistinguishable.

As an important and particularly challenging case the Heston model for parameters that do not satisfy the Feller condition is considered. We modify our parameter test set by decreasing the speed of mean-reversion to $\kappa = 0.5$ which yields $2\kappa\theta/\sigma^2 = 0.64$. When pricing barrier options within a Heston model for which $2\kappa\theta/\sigma^2 \in [0,1]$, the density in (24) exhibits a near-singular behavior in the variance direction (i.e.v_T). This behavior is described in Fang and Oosterlee (2011) for the one-dimensional density of v_T and a logarithmic transformation of the variance domain is proposed as a remedy. We find that this suggestion is effective in our set-up for formula (23) as well. Furthermore, the implementation of the FD scheme is adapted such that a boundary condition is specified for $v = 0$. In Chiarella, Meyer and Kang (2010) the authors suggest a modus operandi for handling the degeneracy on the boundary for $v = 0$ by fitting a quadratic polynomial to barrier prices of surrounding grid points and

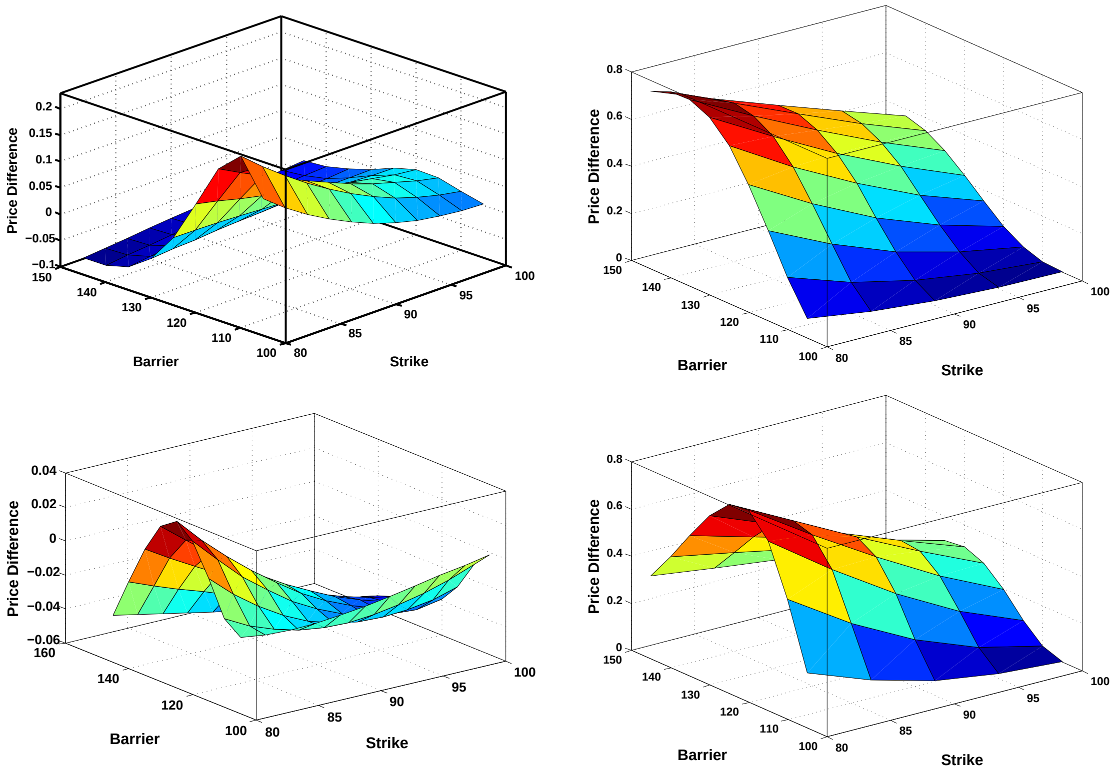

Figure 5: Case C. Upper LHS: X-factor =0.0 (lower) and Z-factor=-2.667 (as before). Upper RHS: X-factor =1.373 (higher) and Z-factor=-2.667 (as before). Lower LHS: X-factor =0.253 (as before) and Z-factor=-1.333 (lower). Lower RHS: X-factor =0.253 (as before) and Z-factor=-13.333 (higher).

subsequently by extrapolating the barrier price for $v = 0$. We consider inserting $v = 0$ in the Heston model price PDE which follows the approach outlined and investigated in Haentjens and In't Hout (2012). On a theoretical basis, it is not straightforward that our presented approach still holds for the case when the Feller condition is not satisfied, however it is interesting to briefly compare the UOC prices to those obtained with the FD scheme.

The price surfaces generated by our approach and by the FD scheme are displayed in figure 8. Zooming into the area where our method appears to perform worst compared to the FD scheme, we find that for up-and-out calls with lower barrier levels of 105, and an initial spot price of 100, the prices range from 0.025 to 1.32 for strike prices between 80 and 100. In that case our method deviates from the results of the FD scheme between 0.005 and 0.032 in absolute terms. Whereas for up-and-out calls with medium barrier levels, $110 - 120$, the prices range from 0.29 to 10.73 for all strike prices and our method deviates from the FD scheme only by 0.033 and 0.22 in absolute terms and by 2% and 11% in relative terms. For up-and-out calls with higher barrier levels, $125 - 145$, the prices range from 3.67 to 21.22 and we observe deviations between 0.5% and 3% in relative terms as shown in figure 7.

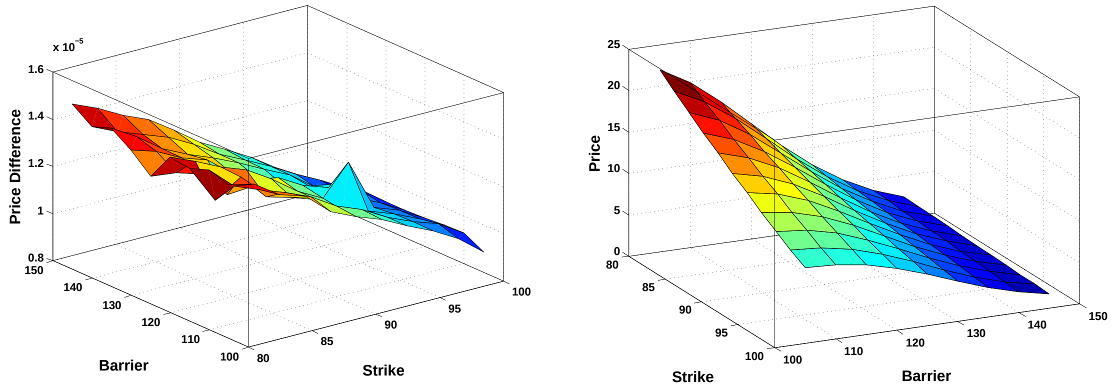

Figure 6: Case C. Differences of absolute up-and-in call option prices between our adjusted approach and a finite differences scheme (LHS), price surfaces of the up-and-in call computed with our approach and a finite differences scheme (RHS).

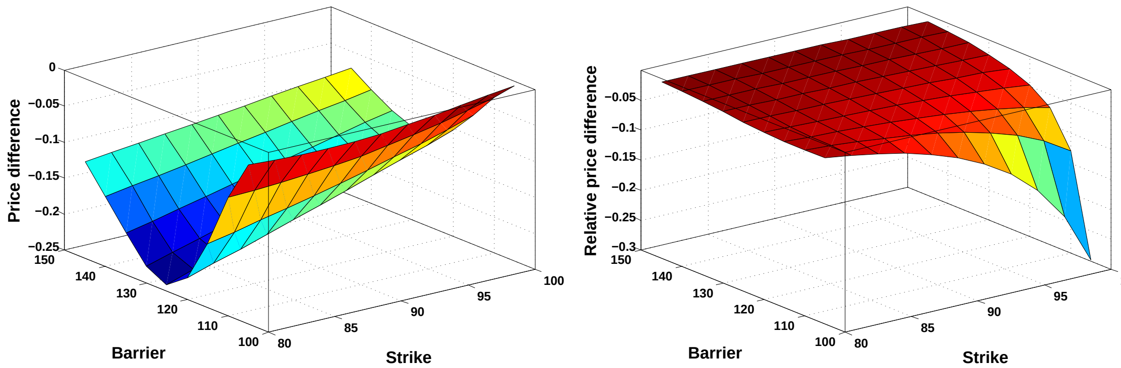

Figure 7: Case C: Differences of absolute and relative up-and-out call option prices between a finite difference scheme and our approach for Heston model parameters that do not fulfill the Feller condition.

7 Conclusion

In this paper, we have studied the pricing of continuously monitored barrier options under the stochastic volatility dynamics of Heston's model. We have derived a semi-analytical solution for barrier options within this model under the assumption of zero correlation and zero interest rate spread. This exact pricing approach was extended to the development of an approximation formula for this type of option for a Heston model with arbitrary interest rate spreads and zero correlation in a first step. In a second step, we have established approximation formulas for barrier options within the general Heston model with arbitrary interest rates, but zero correlation. This is an important special case, because in a (parametric) stochastic volatility model, the volatility skew is generated by a displaced diffusion or a CEV process and the stochastic volatility is often modeled by a Heston model with zero correlation. Finally, we extend our approximation approach for barrier options to the general case

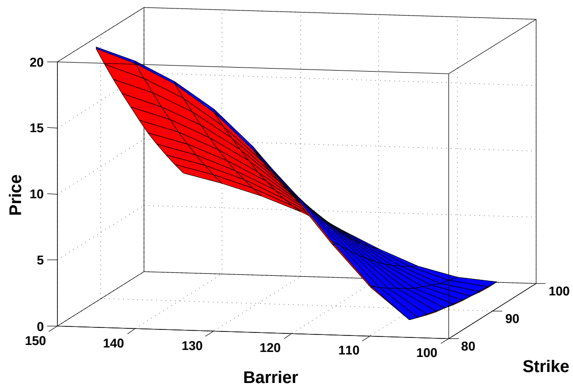

Figure 8: For case C: Price surface for up-and-out call options with low barrier levels computed using a finite difference scheme (blue) and our approach (red) for Heston model parameters that do not fulfill the Feller condition.

of a Heston model with arbitrary interest rates and correlation parameter. All the derivations were carried out using the example of an up-and-out call option. For all three cases, the pricing problem of the up-and-out call is approached by conditioning on the variance paths. Under this knowledge of the variance paths, an expression for the joint density of the logarithmic spot price and its maximum process is derived. This expression is exact for the reduced Heston framework and an approximation of the true joint density for the general Heston model. Hence, when the conditioning is resolved and the pricing problem is solved, we obtain an exact pricing formula or an approximate pricing formula. Generally, the remaining inaccuracies of our developed approximate pricing formula are contributed to the following aspects: the approximation of v_t by a differentiable function and the related measure change, and the error of type 'case B' arising from discarding the conditioning on $\hat{M}$. Numerical examples have demonstrated that the developed pricing method leads to fairly accurate results compared to other conceptually different numerical pricing techniques such as finite difference methods and a formula developed by Lipton (2001) for the case of reduced Heston framework.

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A.1 Appendix

A.1.1 The Reflection Principle

In this section, we show that a driftless Itô process with time-dependent variance, as given in (4), satisfies the reflection principle. A general process $\{Y_t\}_{0\leq t\leq T}$ (for some finite T) of the form

$$Y_t \quad = \quad \int_0^t \beta(s)\, dW_s, \tag{26}$$

is considered, where $\beta(\,\cdot\,)$ is some deterministic[2] continuous function bounded on $[0,T]$. Note that Y is a Gaussian process with heterogeneous instantaneous variance, which distinguishes Y from a Brownian motion. Nevertheless, the first parts of the proof work analogously to their counterparts that deal with Brownian motions:

Lemma 1 *Let Y be defined in (26). Then for all $0 \leq s < t < \infty$ the random variable $Y_t - Y_s$ is independent from $\mathcal{F}_s$.*

Lemma 2 *Let Y be defined in (26) and $u \in \mathbb{R}$. Define*

$$M_t = \exp\Big\{iuY_t + \frac{1}{2}u^2 \int_0^t \beta^2(s)\, ds\Big\}. \tag{27}$$

Then M_t is a martingale.

Lemma 3 *The process Y defined in (26) is strong Markov.*

The proofs are established in the same way as for Brownian motions (see for example Karatzas and Shreve (1991), Theorem 6.15 and Lemma 6.14, Chapter 2. □

Remark 2 *Since a conditional characteristic function determines a conditional distribution, the proof also shows that the distribution of $Y_{\tau+t}$ conditioned on $\mathcal{F}_\tau$ is normal with mean value Y_τ and variance $\int_\tau^{\tau+t} \beta^2(s)\, ds$.*

Remark 3 *A second possibility to show the strong Markov property is by assuming $\beta(t)$ to be continuous in t (which is fulfilled in our application to the Heston model). Then $\{Y_t\}_{t\geq 0}$ is a solution of the SDE $dY_t = \beta(t)\, dW_t$ and since $\beta(t)$ does not depend on Y_t it is also twice continuously differentiable in the space variable. Now theorems can be applied which state the strong Markov property of such solutions (see for example DaPrato (2007), Theorem 8.2).*

[2] In this section, we distinguish between the stochastic process v_t and a deterministic function $\beta(t)$ as the integrand of Y. Although v_t is stochastic in the Heston model setup, we make use of the results established in this section for a process Y with a deterministic integrand by conditioning on the variance paths.

In the derivation of the reflection principle for Brownian motions B, usually a stronger statement than the strong Markov property is proved, i.e. that the increments $B_{\tau+t} - B_\tau$ are independent of $\mathcal{F}_\tau$, for a stopping time τ. This is not true for the process $\{Y_t\}_{t\geq 0}$, since the variance depends on time. Hence, different techniques must be applied to prove the reflection principle for Y. We denote the reflected process (with respect to Y) by

$$\tilde{Y}_t \quad = \quad Y_{\tau\wedge t} - (Y_t - Y_{\tau\wedge t}) \quad t \geq 0 \tag{28}$$

and prove the reflection principle for Y and its supremum.

Theorem 1 *Let $\{Y_t\}_{t\geq 0}$ be an Itô process of the form in equation (26) with deterministic function β and $M_t = \sup_{s\leq t} Y_s$ for $t \geq 0$. Then the reflection principle holds,*

$$\mathbb{P}(M_t \geq x, Y_t < y) = \mathbb{P}(Y_t > 2x - y) \quad \text{for all } t \geq 0,\ x \geq y \vee 0.$$

Proof: The proof is carried out in two steps. First we prove that the process Y defined in (26) and the reflected process $\tilde{Y}$ defined in (28) have the same distribution, i.e. $Y \overset{\mathrm{d}}{=} \tilde{Y}$. Therefore, define the process $Y'_t = Y_{\tau+t} - Y_\tau$. Then Y can be expressed by

$$Y_t \quad = \quad Y_{\tau\wedge t} + Y'_{(t-\tau)^+} = Y_{\tau\wedge t} + \mathbb{1}_{\{t\geq\tau\}} Y'_{t-\tau} + \mathbb{1}_{\{t<\tau\}} Y'_0 = Y_{\tau\wedge t} + \mathbb{1}_{\{t\geq\tau\}} Y'_{t-\tau}$$

and $\tilde{Y}$ by

$$\tilde{Y}_t \quad = \quad Y_{\tau\wedge t} - Y'_{(t-\tau)^+} = Y_{\tau\wedge t} - \mathbb{1}_{\{t\geq\tau\}} Y'_{t-\tau} - \mathbb{1}_{\{t<\tau\}} Y'_0 = Y_{\tau\wedge t} - \mathbb{1}_{\{t\geq\tau\}} Y'_{t-\tau}.$$

Note that, the random variables Y'_s are defined for $s \geq -\tau$. Since $Y'_{t-\tau} = Y_t - Y_\tau$, $\{\tau \leq t\} \in \mathcal{F}_\tau$ and

$$\exp\left(iu(Y_t - Y_\tau)\mathbb{1}_{\{\tau\leq t\}}\right) = 1 - \mathbb{1}_{\{\tau\leq t\}} + \mathbb{1}_{\{\tau\leq t\}} \exp\left(iu(Y_t - Y_\tau)\right),$$

we have almost surely that

$$\begin{aligned} \mathbb{E}\left[\exp\left(iuY'_{(t-\tau)^+}\right) \mid \mathcal{F}_\tau\right] &= 1 - \mathbb{1}_{\{\tau\leq t\}} + \mathbb{1}_{\{\tau\leq t\}} \mathbb{E}[\exp\left(iu(Y_t - Y_\tau)\right) \mid \mathcal{F}_\tau] \\ &= 1 - \mathbb{1}_{\{\tau\leq t\}} + \mathbb{1}_{\{\tau\leq t\}} \mathbb{E}[\exp\left(-iu(Y_t - Y_\tau)\right) \mid \mathcal{F}_\tau] \\ &= \mathbb{E}\left[\exp\left(iu(-Y'_{(t-\tau)^+})\right) \mid \mathcal{F}_\tau\right], \end{aligned} \tag{29}$$

where in the second equation we used the fact that $(Y_t - Y_\tau)$ conditioned on $\mathcal{F}_\tau$ is normally distributed with mean zero for $t > \tau$ (justified by lemma 3 and remark 2 in the appendix A.1.1).
Because the equality in (29) is true for all $t \geq 0$ we conclude that

$$\begin{aligned} \mathbb{E}\left[\exp\left(iuY'_{(t-\tau)^+}\right)\right] &= \mathbb{E}\left[\mathbb{E}\left[\exp\left(iuY'_{(t-\tau)^+}\right) \mid \mathcal{F}_\tau\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\exp\left(iu(-Y'_{(t-\tau)^+})\right) \mid \mathcal{F}_\tau\right]\right] \\ &= \mathbb{E}\left[\exp\left(iu(-Y'_{(t-\tau)^+})\right)\right]. \end{aligned}$$

The first part of the assertion follows thereafter. Secondly, using the first part of the proof, $Y \stackrel{d}{=} \tilde{Y}$, it immediately follows that

$$\mathbb{P}(M_t \geq x, Y_t \leq y) \quad = \quad \mathbb{P}(\tilde{Y}_t \geq 2x - y) = \mathbb{P}(Y_t \geq 2x - y).$$

□

A.1.2 Appendix for the Second Step

Proposition 3 *For the case $r_d = r_f$ and $\rho = 0$. The random variable $(\hat{X}_T, \hat{Y}_T)$ is normally distributed with zero mean and covariance matrix*

$$\Sigma \quad = \quad \begin{pmatrix} \nu^2_{\text{inv}}(T) & T \\ T & \nu^2(T) \end{pmatrix}.$$

Proof: The characteristic function of $(\hat{X}_T, \hat{Y}_T)$ is given by

$$\begin{aligned} \mathbb{E}^v \left[\exp\left\{ iu_1 \hat{X}_T + iu_2 \hat{Y}_T \right\} \right] &= \mathbb{E}^v \left[\exp\left\{ i \int_0^T \left(u_1 \frac{1}{\sqrt{v_s}} + u_2 \sqrt{v_s} \right) d\hat{W}_s \right\} \right] \\ &= \exp\left\{ -\frac{1}{2} (u_1^2 \int_0^T \frac{1}{v_s} ds + 2u_1 u_2 T + u_2^2 \int_0^T v_s ds) \right\}. \end{aligned}$$

□

Proposition 4 *The following assertions are proved as in proposition 3 and are derived from the theory of normal distributions.*

- *For the case r_d, r_f arbitrary and $\rho = 0$. The random variable $(\hat{X}_T, \kappa_1 \hat{Y}_T)$ is normally distributed with zero mean and covariance matrix*

$$\Sigma \quad = \quad \begin{pmatrix} \nu^2_{\text{inv}}(T) & \frac{T^2}{\nu^2(T)} \\ \frac{T^2}{\nu^2(T)} & \frac{T^2}{\nu^2(T)} \end{pmatrix},$$

  hence ε is normal with zero mean and variance $\sigma^2_\varepsilon = \nu^2_{\text{inv}}(T) - \frac{T^2}{\nu^2(T)}$.

- *For the case r_d, r_f arbitrary and arbitrary ρ. The random variable $(\hat{X}_T, \hat{Y}_T, \hat{Z}_T)$ is normally distributed with zero mean and covariance matrix*

$$\Sigma \quad = \quad \begin{pmatrix} \rho_2^2 \nu^2_{\text{inv}}(T) & \rho_2^2 T & \rho_2^2 \nu^2_{II}(T) \\ \rho_2^2 T & \rho_2^2 \nu^2(T) & \rho_2^2 (\nu'(T))^2 \\ \rho_2^2 \nu^2_{II}(T) & \rho_2^2 (\nu'(T))^2 & \rho_2^2 \nu^2_I(T) \end{pmatrix}.$$

- *For the case r_d, r_f arbitrary and arbitrary ρ. The random variable $(\varepsilon^X_T, \varepsilon^Z_T)$ is normally distributed with zero mean and covariance matrix*

$$\Sigma \quad = \quad \begin{pmatrix} \rho_2^2 \nu^2_{\text{inv}}(T) - \kappa_1 T \rho_2^2 & \rho_2^2 \nu^2_{II}(T) - \kappa_3 T \rho_2^2 \\ \nu^2_{II}(T) \rho_2^2 - \kappa_3 T \rho_2^2 & \rho_2^2 \nu^2_I(T) - \kappa_3 \rho_2^2 (\nu'(T))^2 \end{pmatrix},$$

  and is independent from $\hat{Y}_T$.

A.1.3 Appendix for the Fourth Step

The bivariate characteristic function of v_T and $\int_0^T v_t dt$ is defined by

$$\varphi(u,w) = \mathbb{E}\left[\exp\left(iu\int_0^T v_t dt + iwv_T\right)\right]. \tag{30}$$

By applying the Feynman-Kac formula and solving the resulting partial differential equation, the expectation has the following solution

$$\mathbb{E}\left[\exp\left(iwv_T + iu\int_0^T v_t dt\right)\right] = \exp\left[A(T,u,w)v_0 + B(T,u,w)\right]$$

with $d = \sqrt{\kappa^2 - 2\sigma^2 iu}$, $e^{\pm} = 1 \pm \exp(-d\tau)$, $\gamma = de^+ + (\kappa - \sigma^2 iw)e^-$ and

$$\begin{aligned} A(\tau,u,w) &= \frac{-(-2iu + \kappa iw)e^- + diwe^+}{\gamma} \\ B(\tau,u,w) &= \frac{\kappa\theta}{\sigma^2}(\kappa - d)\tau + \frac{2\kappa\theta}{\sigma^2}\ln\frac{2d}{\gamma}. \end{aligned}$$

The joint density of v_T and $\nu^2(T) = \int_0^T v_s ds$ is given by the double Fourier inversion of the characteristic function in equation (30)

$$d_{v_T,\nu^2(T)}(x,y) = \frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} e^{-iux-iwy}\varphi(u,w)dwdu.$$

In Chiarella and Ziogas (2005), using PDE methods the authors show that the joint density of v_T and $\nu^2(T)$ has a closed-form solution with respect to v_T. Here, we follow the approach described in Griebsch (2011) using integration techniques to derive this representation of the joint density, which yields

$$\begin{aligned} d_{v_T,\nu^2(T)}(x,y) &= \frac{1}{2\pi}\exp\left(L\kappa\tau + (v_0 - y)\frac{\kappa}{\sigma^2}\right) \\ &\quad \times \int_{\mathbb{R}} \frac{2d}{\sigma^2 e^-} e^{-iux - Ld\tau - (v_0+y)\frac{de^+}{\sigma^2 e^-}} \left(\frac{ye^{d\tau}}{v_0}\right)^{L-\frac{1}{2}} I_{2L-1}\left(\frac{4d}{\sigma^2 e^-}\sqrt{yv_0 e^{-d\tau}}\right) du, \end{aligned}$$

with $L = \frac{\kappa\theta}{\sigma^2}$ and where I denotes the modified Bessel function for complex arguments.